Labelled Markov Processes as Generalised Stochastic Relations

Michael Mislove [a,1], Dusko Pavlovic [c,2] and James Worrell [b]

[a] *Tulane University, Department of Mathematics, New Orleans, USA*

[b] *Computing Laboratory, University of Oxford, UK*

[c] *Kestrel Institute, Palo Alto, USA*

Abstract

Labelled Markov processes (LMPs) are labelled transition systems in which each transition has an associated probability. In this paper we present a universal LMP as the spectrum of a commutative C^*-algebra consisting of formal linear combinations of labelled trees. This yields a simple *trace-tree* semantics for LMPs that is fully abstract with respect to probabilistic bisimilarity. We also consider LMPs with distinguished entry and exit points as stateful stochastic relations. This allows us to define a category **LMP**, with measurable spaces as objects and LMPs as morphisms. Our main result in this context is to provide a predicate-transformer duality for **LMP** that generalises Kozen's duality for the category **SRel** of stochastic relations.

Key words: Labelled Markov process, Stochastic relation, Probabilistic bisimulation, Stone duality, C^*-algebra, Comonad.

1 Introduction

Probabilistic models are important for capturing quantitative aspects of process behaviour, such as performance and reliability, e.g., the average response time to a given action, or the probability with which a failure occurs. For this reason there has been extensive research into adapting the concepts and results of classical concurrency theory to the probabilistic case. In particular,

[1] Supported by the Office of Naval Research, grant No. N000149910150 and the National Science Foundation, grant No. CCR-0208743.

[2] Supported by the EHS and SGER programs of the National Science Foundation, under contract No. CCR-0209-004 and CCR-0345-397.

the notion of bisimilarity has been adapted to probabilistic systems [17,9,16], and its equational theory investigated in [22,4] among many others.

This paper is concerned with the semantics of certain probabilistic labelled transition systems, called *labelled Markov processes (or LMPs)* [9,11,7]. These can be seen as coalgebras of an endofunctor $X \mapsto \mathcal{M}(\mathrm{Act} \times X)$ on the category **Mes** of measurable spaces, where Act is a set of actions and $\mathcal{M}(\mathrm{Act} \times X)$ is the space of all subprobability measures on $\mathrm{Act} \times X$. The coalgebra homomorphisms yield a natural notion of maps between LMPs, called *zig-zag maps* in [9].

Our first contribution is to construct a *universal LMP*. The universal property here is finality: we construct an LMP that is final in the category of LMPs and zig-zag maps. Such a universal LMP has previously been constructed as the solution of a domain equation [11,7]. Here we exploit Stone duality for real commutative C^*-algebras to characterise the universal LMP as the spectrum of a C^*-algebra generated by a class of *trace trees*. These trace trees are closely related to the tests introduced by Larsen and Skou [17] in their paper characterising bisimilarity as a testing equivalence. A trace tree is essentially a finite Act-labelled tree, that is, a trace with branching. Adding algebraic and order-theoretic structure transforms the set of trace trees into a preordered, commutative ring, which can then be completed relative to a natural semi-norm into a commutative C^*-algebra. The spectrum of this C^*-algebra forms the state space of the universal LMP. An important consequence of this characterisation—one of our main results—is that two LMPs are bisimilar iff they have the same probability of performing each trace tree.

A second contribution of this paper involves generalising the notion of labelled Markov process to accommodate *interfaces*. We do this by specifying for each LMP a measurable space of *entry points* and a measurable space of *exit points*. A similar extension of labelled transition systems occurs in the work of Bloom and Esik [6] in the context of iteration theories, and in the notion of charts, introduced by Milner [18]. Thus we obtain a category **LMP** whose objects are measurable spaces and in which a morphism $X \to Y$ is an LMP with entry points X and exit points Y. (This should not be confused with the category of LMPs and zig-zag maps, in which LMPs are the objects.) **LMP** includes the category **SRel** of stochastic relations [3,20] as a subcategory: stochastic relations can be seen as stateless LMPs. Our main result in this context is to characterise the dual of **LMP** as the co-Kleisli category of certain comonad in the category of ordered rings.

Our duality for **LMP** extends Kozen's [14] duality for **SRel**. According to the latter, the dual of a stochastic relation $X \to Y$ is a monotone linear map $B(Y) \to B(X)$, where $B(X)$ denotes the ordered vector space of bounded real-valued measurable functions on X with the pointwise order. In fact, a stochastic relation $X \to Y$ is a measurable map $\mu : X \to \mathcal{M}(Y)$, and the dual

	Morphism	Dual
SRel	$X \to \mathcal{M}(Y)$	$B(Y) \to B(X)$ [monotone linear map]
LMP	$X + S \longrightarrow \mathcal{M}(Y + (\mathrm{Act} \times S))$	$\mathcal{T}B(Y) \to B(X)$ [monotone ring map]

Fig. 1. Dualities for **SRel** and **LMP**.

map $\widehat{\mu} : B(Y) \to B(X)$ is defined by $\widehat{\mu}(f)(x) = \int_Y f \, d\mu_x$.

Kozen's duality underlies a predicate-transformer semantics for an imperative programming language with probabilistic choice. In this view predicates are measurable functions on the set of exit points. However our development is in the context of interactive processes rather than imperative programs. Correspondingly, our class of predicates is richer than Kozen's. Given an LMP with set of exit points Y, the relevant predicates are trace trees trees whose leaves are labelled by elements of $B(Y)$. These trace trees generate a preordered ring that we call $\mathcal{T}B(Y)$. Then the dual of an LMP $\mathcal{S} : X \to Y$ is a monotone ring map $\mathcal{T}B(Y) \to B(X)$. We show also that $\mathcal{T}$ is a comonad on the category of preordered rings, so that the dual of an LMP is a map in the co-Kleisli category of $\mathcal{T}$. We further show that composition of LMPs corresponds to co-Kleisli composition on the dual side.

The situation is summarised in Figure 1 which shows that the addition of state to stochastic relations corresponds to adding a comonad on the dual side. It is also noteworthy that for **SRel** the dual maps preserve addition in $B(Y)$, whereas for **LMP** the dual maps preserve both addition and multiplication in $\mathcal{T}B(Y)$. There is no contradiction here; while $\mathcal{T}B(Y)$ is in some sense generated by $B(Y)$, only the additive structure of $B(Y)$ is preserved in $\mathcal{T}B(Y)$. Thus every monotone additive map $B(Y) \to B(X)$ extends to a monotone ring map $\mathcal{T}B(Y) \to B(X)$. However the multiplicative structure of $\mathcal{T}B(Y)$ plays an important role. Intuitively it reflects the fact that we consider LMPs modulo bisimilarity, and bisimilarity is a branching-time equivalence.

Simplified versions of the results in this paper were first described in the extended abstract [19].

2 Labelled Markov Processes

In this section we formally define the class of probabilistic transition systems that we study in this paper: labelled Markov processes (LMPs). Our notion of LMP extends that of [9] by specifying sets of entry and exit points. This extension allows us to define composition of LMPs. The resulting category of LMPs includes the category **SRel** of stochastic relations as a subcategory, where stochastic relations can be seen as stateless LMPs. The connection with stochastic relations will be explored in the next section.

Given a measurable space $X = (X, \Sigma_X)$ consisting of a set X and a σ-field Σ_X of subsets of X, we write $\mathcal{M}X$ for the set of subprobability measures on X. For each measurable subset $A \subseteq X$ we have an evaluation function $p_A : \mathcal{M}X \to [0, 1]$ sending μ to μA. Then $\mathcal{M}X$ becomes a measurable space by giving it the smallest σ-field such that all the evaluations p_A are measurable. (In fact, this is the smallest σ-field such that integration against any measurable function $g : X \to [0, 1]$ yields a measurable map $\int g\, d{-} \colon \mathcal{M}X \to [0, 1]$.) Next, $\mathcal{M}$ is turned into an endofunctor on the category **Mes** of measurable spaces by defining $\mathcal{M}(f)(\mu) = \mu \circ f^{-1}$ for $f : X \to Y$ measurable and $\mu \in \mathcal{M}X$.

Theorem 2.1 (Giry [12]) *The functor* $\mathcal{M} : \mathbf{Mes} \to \mathbf{Mes}$ *defines a monad on* **Mes***; the unit is given by* $\eta_X(x) = \delta_x$ *and the multiplication* $\mu : \mathcal{M}^2 \stackrel{\cdot}{\longrightarrow} \mathcal{M}$ *is given by integration.*

Henceforth we assume a fixed finite set Act of actions or events.

Definition 2.2 *Given measurable spaces* X *and* Y*, a* ***labelled Markov process*** $\mathcal{S} \colon X \to Y$ *is a pair* (S, μ) *consisting of a measurable space* S *and a measurable map* $\mu : X + S \to \mathcal{M}(Y + (\mathrm{Act} \times S))$.

We think of X and Y as the interfaces of $\mathcal{S}$, where X is the space of entry points and Y is the space of exit points, and we think of S as the state space. Given $s \in S$ and $a \in \mathrm{Act}$, $\mu_s(\{a\} \times E)$ is the probability that the process in state s makes an a-transition to a measurable set of states $E \subseteq S$. Similarly if $E \subseteq Y$ is a measurable set of exit points, then $\mu_s(E)$ is the probability that state s makes a transition to the set E. Note that μ_s is a *sub-probability* distribution on $(\mathrm{Act} \times S) + Y$. We interpret the difference between the total mass of μ_s and 1 as the probability of deadlock. We also adopt the notation $\mu_{s,a}$ for the subprobability measure on S given by $\mu_{s,a}(E) = \mu_s(\{a\} \times E)$, and we write $\mu_{s,\varepsilon}$ for the subprobability measure on Y given by $\mu_{s,\varepsilon}(E) = \mu_s(E)$. Thus transitions to exit points are thought of as ε-transitions.

Next we generalise the notion of zig-zag maps between LMPs [9] to the case with entry and exit points.

Definition 2.3 *Let* $\mathcal{S}, \mathcal{S}' : X \to Y$ *be LMPs, where* $\mathcal{S} = (S, \mu)$ *and* $\mathcal{S}' = (S', \mu')$*. A function* $h \colon S \to S'$ *between their respective state spaces is a zig-zag map if the following diagram commutes.*

$$
\begin{tikzcd}
X+S \arrow[r, "\mu"] \arrow[d, "id_X+h"'] & \mathcal{M}(Y+(\mathrm{Act}\times S)) \arrow[d, "\mathcal{M}(id_Y+(id_{\mathrm{Act}}\times h))"] \\
X+S' \arrow[r, "\mu'"'] & \mathcal{M}(Y+(\mathrm{Act}\times S'))
\end{tikzcd}
$$

The commuting of this diagram is equivalent to the following two conditions, where g is the function $id_X + h$:

- $\mu_{s,a}(h^{-1}(E)) = \mu'_{g(s),a}(E)$ *for all* $s \in X+S$*, measurable* $E \subseteq S'$ *and* $a \in \mathrm{Act}$*.*
- $\mu_{s,\varepsilon}(E) = \mu'_{g(s),\varepsilon}(E)$ *for all* $s \in X+S$ *and measurable* $E \subseteq Y$*.*

Note that we only define zig-zag maps between LMPs with the same sets of entry points and exit points (see below).

$$
\begin{tikzcd}
X \arrow[r, bend left, "\mathcal{S}"] \arrow[r, bend right, "\mathcal{S}'"'] \arrow[r, phantom, "\Downarrow h"] & Y
\end{tikzcd}
$$

This suggests that zig-zag maps could be seen as 2-cells in a bicategory whose 0-cells are measurable spaces and whose 1-cells are LMPs. However we do not pursue this idea; rather we use zig-zag maps to define a notion of *bisimulation equivalence* between LMPs, and we focus on the resulting (genuine) category of measurable spaces and equivalence classes of LMPs.

2.1 Probabilistic Bisimulation

Probabilistic bisimulation was introduced by Larsen and Skou [17] as a probabilistic analog of strong bisimulation for labelled transition systems. They defined a probabilistic bisimulation on an LMP (with countable state space) to be an equivalence relation on the state space such that equivalent states have the same probability of transitioning to each equivalence class under a given action. This relational definition was extended to LMPs with non-discrete state spaces in [9]. However, in this paper it will be more technically convenient to work with an alternative formulation of a bisimulation as a cospan of zig-zag maps [8].

Definition 2.4 *Let $\mathcal{S}, \mathcal{S}' \colon X \to Y$ be LMPs. We say that $\mathcal{S}$ and $\mathcal{S}'$ are* ***bisimilar*** *if there exists a third LMP $\mathcal{S}'' \colon X \to Y$ and zig-zag maps $h \colon \mathcal{S} \to \mathcal{S}''$ and $g \colon \mathcal{S}' \to \mathcal{S}''$.*

Note that the entry points of $\mathcal{S}$ and $\mathcal{S}'$ are identified by g and h.[3] Intuitively, Definition 2.4 captures the idea that $\mathcal{S}$ and $\mathcal{S}'$ are indistinguishable at each entry point $x \in X$.

[3] Strictly speaking we should say that the entry points of $\mathcal{S}$ and $\mathcal{S}'$ are identified in $\mathcal{S}''$ by $id_X + g$ and $id_X + h$.

3 LMPs as Generalised Stochastic Relations

In this section we define the category **SRel** of stochastic relations and its stateful generalisation the category **LMP** of LMPs. We also summarise Kozen's duality for **SRel** in anticipation of its later generalisation to **LMP**.

Definition 3.1 *The category* ***SRel*** *of stochastic relations is the Kleisli category of the Giry monad. Thus a stochastic relation $f : X \to Y$ is a measurable function $f : X \to \mathcal{M}(Y)$.*

The composite of stochastic relations $f : X \to Y$ and $g : Y \to Z$ is given by

$$(g \circ f)(x)(C) = \int_Y g(\cdot)(C)\, df_x\,,$$

where $x \in X$, $C \in \Sigma_Z$, and f_x denotes the measure $f(x)$ on Y. Identities in **SRel** are given by point measures: $id_X : X \to X$ is defined by $id_X(x) = \delta_x$ where

$$\delta_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Note that coproducts lift from **Mes** to **SRel**. In particular, the binary coproduct of X and Y in **SRel** is the disjoint sum $X + Y$, with injections $inl : X \to X + Y$ and $inr : Y \to X + Y$ given by $inl(x) = \delta_x$ and $inr(y) = \delta_y$.

Next we describe Kozen's [14] duality between stochastic relations and linear maps.

Definition 3.2 *The category* ***SPT*** *of* ***stochastic predicate transformers*** *has as objects measurable spaces. To such a measurable space X we can associate the ordered vector space $\mathcal{B}(X)$ of bounded, real-valued measurable functions on X, endowed with the pointwise order. A morphism $X \to Y$ in* ***SPT*** *is then a linear, monotone function $\varphi : \mathcal{B}(X) \to \mathcal{B}(Y)$ satisfying $\varphi(1) \leq 1$.*

Theorem 3.3 (Kozen [14]) *The category* ***SRel*** *is dually equivalent to the category* ***SPT*** *under the correspondence that associates to $h : X \to Y$ in* ***SRel*** *the mapping from $\varphi : \mathcal{B}(Y) \to \mathcal{B}(X)$, where*

$$\varphi(f)(x) = \int_Y f\, dh_x\,,$$

and to $\varphi : \mathcal{B}(Y) \to \mathcal{B}(X)$ the ***SRel*** *morphism $h : X \to Y$, where*

$$h(x)(A) = \varphi(\chi_A)(x)\,.$$

As we shall see later, our main theorem gives a stateful generalisation of this duality.

3.1 The Category ***LMP***

In this subsection we extend **SRel** to a category **LMP** whose objects are measurable spaces and whose morphisms are (bisimulation-equivalence classes of) LMPs. Given measurable spaces X and Y, an LMP $\mathcal{S} : X \to Y$ represents a morphism from X to Y in **LMP**; another LMP $\mathcal{S}' : X \to Y$ represents the same morphism iff $\mathcal{S}$ and $\mathcal{S}'$ are bisimilar. Comparing Definitions 2.2 and 3.1, we observe that a stochastic relation $X \to Y$ can be regarded as LMP with empty state space. It is also clear from Definition 2.4 that two stochastic relations $X \to Y$ are bisimilar qua LMPs iff they are identical. Thus stochastic relations are morphisms in **LMP**.

Next we define composition in **LMP**. We define composition of LMPs (rather than of equivalence classes) following the composition-as-integration pattern for stochastic relations. Proposition 3.4 then shows that this lifts to a well-defined composition in **LMP**. Let $\mathcal{S} \colon X \to Y$ and $\mathcal{S}' \colon Y \to Z$ be LMPs with $\mathcal{S} = (S, \mu)$ and $\mathcal{S}' = (S', \mu')$. Intuitively, the composition $(\mathcal{S}' \circ \mathcal{S}) : X \to Z$ is obtained by connecting the exits of $\mathcal{S}$ with the entries of $\mathcal{S}'$. Formally $\mathcal{S}' \circ \mathcal{S} = (S + S', \rho)$, where the transition measure ρ is given by

$$\rho_s(B) = \begin{cases} \mu_s(B) & \text{if } s \in S, B \subseteq \mathrm{Act} \times S \\ \int_Y \mu'_{(\cdot)}(B) d\mu_s & \text{if } s \in S,\ B \subseteq (\mathrm{Act} \times S') + Z \\ \mu'_s(B) & \text{if } s \in S', B \subseteq (\mathrm{Act} \times S') + Z \\ 0 & \text{if } s \in S', B \subseteq \mathrm{Act} \times S\,. \end{cases}$$

Proposition 3.4 *Composition in* ***LMP*** *is well-defined and associative. The identity maps and coproducts in* ***LMP*** *are inherited from* ***SRel****.*

The proof of Proposition 3.4 is routine. However we will give an indirect proof later as an application of our duality for **LMP**.

4 Stone Duality for C^*-Algebras

This section contains some background definitions and results about preordered rings and C^*-algebras from the monograph of Johnstone [15].

Let A be a commutative ring with identity 1. Since we are primarily interested in rings of functions, we use f, g to denote typical elements of A. As is usual, given $n \in \mathbb{N}$, we write $n \in A$ for the n-fold sum of the identity. We say that A is a *preordered ring* if it is equipped with a preorder satisfying

- $0 \sqsubseteq f^2$ (all squares are positive)
- $f \sqsubseteq f'$ implies $f + g \sqsubseteq f' + g$
- $f \sqsubseteq f'$ and $0 \sqsubseteq g$ implies $f \cdot g \sqsubseteq f' \cdot g$.

Equivalently we can define such a preorder by specifying a set $P \subseteq A$ that is closed under addition and multiplication, and which contains all squares. Such a set is called a *positive cone* in A. Then a preorder on A is defined by $f \sqsubseteq g$ iff $g - f \in P$.

We denote by **ORng** the category of preordered rings and monotone ring homomorphisms.

We say that a preordered ring A is *Archimedean* if for all f there exists a positive integer n with $f \sqsubseteq n$. If the additive group of A is torsion-free and divisible, so that A admits a $\mathbb{Q}$-algebra structure, then we may define a seminorm on A by

$$||f|| = \inf\{q \in \mathbb{Q} : -q \sqsubseteq f \sqsubseteq q\}. \tag{1}$$

(Here if $q = n/m \in \mathbb{Q}$, then we let q denote the unique element of $q \in A$ satisfying $m \cdot q = n$.) This seminorm satisfies

$$||f + g|| \leq ||f|| + ||g|| \text{ and } ||f \cdot g|| \leq ||f||\, ||g||\,.$$

However we may have $||f|| = 0$ for nonzero f, that is, we have a seminorm rather than a norm.

Definition 4.1 *A partially ordered ring A is a* ***real C^*-algebra*** *if*

- *the additive group of A is torsion free and divisible, and*
- *Equation 1 defines a norm with respect to which A is complete.*

The category $\mathbf{C^*}$-**Alg** *is the full subcategory of* **ORng** *determined by the class of C^*-algebras. .*

Here we should emphasise that we work with the notion of *real* C^*-algebras as opposed to the more widely known notion of *complex* C^*-algebras (cf. Naimark [13, Theorem III.2.1]). Also we recall from [15, Lemma 4.5] that an element of a C^*-algebra is positive iff it is a square. Thus the partial order is determined by the ring structure, and ring homomorphisms between C^*-algebras are automatically order preserving.

Example 4.2 *Let Y be a measurable space and $B(Y)$ the set of bounded measurable real-valued functions on Y equipped with the pointwise order. Then $B(Y)$ is a C^*-algebra. The induced norm is here is just the supremum norm, and $B(Y)$ is complete in this norm since the pointwise limit of a sequence of measurable functions is again measurable.*

Definition 4.3 *A* ***character*** *of a C^*-algebra A is a ring homomorphism $\varphi\colon A \to \mathbb{R}$. The* ***spectrum*** *of A, denoted* $\operatorname{Spec} A$*, is the space of characters of A in the* ***Zariski topology****, which is generated by the cozero sets* $\operatorname{coz}(f) = \{\varphi : \varphi(f) \neq 0\}$ *where $f \in A$.*

The spectrum of a C^*-algebra is a compact Hausdorff space. Conversely, the ordered ring $C(X)$ of continuous real-valued functions on a compact Hausdorff space X is a C^*-algebra. This association of compact Hausdorff spaces and C^*-algebras is functorial, and yields a dual equivalence:

Theorem 4.4 (Stone) *The category* **KHaus** *of compact Hausdorff spaces and continuous maps is dually equivalent to* $\mathbf{C^*}$**-Alg**.

5 Trace Trees

Fix a measurable space Y of exit points. We define a grammar of *trace trees*, generated from the set $B(Y)$ of bounded measurable real-valued functions on Y by *prefixing* and *multiplication*. These trace trees are simplified versions of the tests considered by Larsen and Skou [17] in their paper characterising bisimulation as a testing equivalence, but adapted to the fact that we consider LMPs with exit points.

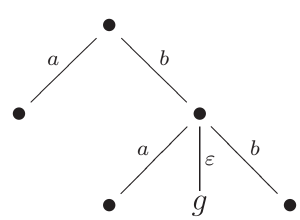
The trace trees are given by the grammar

$$t ::= 1 \mid \varepsilon.g \mid a.t \mid t * t\,, \tag{2}$$

where $g \in B(Y)$ and $a \in \mathrm{Act}$.

We think of 1 as the null trace; $a.t$ is read as t *prefixed by* $a \in \mathrm{Act}$; $\varepsilon.g$ is read as g prefixed by the silent action ε; finally we call $t_1 * t_2$ the product of t_1 and t_2. Note the distinction between prefixing and product. We adopt the convention that prefixing binds more tightly than product. We also sometimes elide the symbol 1 when denoting non-trivial trace trees, e.g., we write $a * b.c$ for $a.1 * b.c.1$.

We call the terms generated by (2) trace trees because there is a very natural way to view them as trees whose edges are labelled in $\mathrm{Act} \cup \{\varepsilon\}$ and whose leaves are either unlabelled or labelled by elements of $B(Y)$. For instance, the term $a.1 * b.((a.1 * \varepsilon.g) * b.1)$ is pictured as



Definition 5.1 specifies the probability $t_{\mathcal{S}}(s)$ that an LMP $\mathcal{S}$ in state s can perform the trace tree t. The null trace is performed with probability one in any state. The probability of performing $a.t$ is the probability of performing an a-action and then doing t. Prefixing by ε is interpreted similarly. For instance, if $g = \chi_A$ is the characteristic function of a measurable set $A \subseteq Y$, then the probability of doing $\varepsilon.g$ is the probability of making an ε-transition

to a state in A. Finally, the probability that a state performs $t_1 * t_2$ is the product of the probability it performs t_1 and the probability it performs t_2.

Definition 5.1 *Given an LMP $\mathcal{S} : X \to Y$, where $\mathcal{S} = (S, \mu)$, each trace tree t is interpreted as a real-valued function $t_{\mathcal{S}}$ on $S + X$ by:*

$$\begin{aligned} 1_{\mathcal{S}}(s) &= 1 \\ (a.t)_{\mathcal{S}}(s) &= \int_S t_{\mathcal{S}}\, d\mu_{s,a} \\ (\varepsilon.g)_{\mathcal{S}}(s) &= \int_S g\, d\mu_{s,\varepsilon} \\ (t_1 * t_2)_{\mathcal{S}}(s) &= (t_1)_{\mathcal{S}}(s) \cdot (t_2)_{\mathcal{S}}(s)\,. \end{aligned}$$

Without product, the grammar (2) would just specify a language of traces, and $t_{\mathcal{S}}(s)$ would give the probability that state s can perform trace t. Product is required to discriminate between processes that are trace equivalent but not bisimilar. We refer the reader to Larsen and Skou [17] and Abramsky [1] for further discussion about related classes of branching traces (or tests), both in the context of probabilistic and nondeterministic labelled transition systems.

The following theorem, our first main result, states that LMPs are characterised up to bisimilarity by their trace-tree semantics. The proof, which will be given later, relies on an application of Stone duality for real C^*-algebras.

Theorem 5.2 *Two LMPs $\mathcal{S}, \mathcal{S}' : X \to Y$ are bisimilar iff $t_{\mathcal{S}}(x) = t_{\mathcal{S}'}(x)$ for all trace trees t and $x \in X$.*

This result generalises and simplifies a result of Larsen and Skou [17]. Our class of tests is a simplification of theirs, and their result applied to LMPs with a strong discreteness assumption. Also the proofs are quite different: [17] use statistical arguments, including Chebyshev's inequality.

6 A Comonad of Trace Trees

In this section, we define a comonad $(\mathcal{T}, \xi, \delta)$ on the category **ORng** based on a generalised notion of trace tree. Given a preordered ring R, we present a new ring $\mathcal{T}(R)$ by generators and relations, where the set of generators is given by the following grammar (which corresponds to (2), but with $B(Y)$ replaced by an arbitrary ring R):

$$t ::= 1 \mid \varepsilon.r \mid a.t \mid t * t,$$

where $a \in \mathrm{Act}$ and $r \in R$.

The terms generated by the above grammar are called the *trace trees over* R; thus our original notion of trace tree in Section 5 gives the trace trees over $B(Y)$. For each trace tree t we include a generator $[t]$ in the presentation of

$\mathcal{T}(R)$. (We distinguish between trace trees and the corresponding generators in the interests of clarity, but we will later drop the distinction.) The relations we postulate in the presentation of $\mathcal{T}(R)$ include the following equations, where $r_1, r_2 \in R$ and t_1, t_2 are trace trees.

$$[\varepsilon.0] = 0 \tag{3}$$
$$[\varepsilon.r_1] + [\varepsilon.r_2] = [\varepsilon.(r_1 + r_2)] \tag{4}$$
$$[1] = 1_{\mathcal{T}(R)} \tag{5}$$
$$[t_1 * t_2] = [t_1] \cdot [t_2] \tag{6}$$

Intuitively, Equations 3 and 4 say that prefixing by ε is linear. Equation 5 says that the null tree 1 is interpreted as the multiplicative identity in $\mathcal{T}(R)$. Lastly, Equation 6 says that the product operation $*$ on trace trees corresponds to multiplication in $\mathcal{T}(R)$ (which is denoted $\cdot$).

We define the preorder on $\mathcal{T}(R)$ to be the least one satisfying the axioms for a preordered ring (cf. Section 4), plus the following clauses in which $r_1, r_2 \in R$ and t_1, t_2 are trace trees over R.

$$[t_1] \sqsubseteq [t_2] \implies [a.t_1] \sqsubseteq [a.t_2] \tag{7}$$
$$r_1 \sqsubseteq r_2 \implies [\varepsilon.r_1] \sqsubseteq [\varepsilon.r_2] \tag{8}$$
$$[\varepsilon.1_R] + \sum_{a \in \mathrm{Act}} [a.1] \sqsubseteq 1_{\mathcal{T}(R)} \,. \tag{9}$$

These inequalities are connected with our interpretation of prefixing as integration against a positive measure. Inequalities 7 and 8 say that prefixing by $a \in \mathrm{Act}$ or by ε is monotone, whereas Inequality 9 is connected with the fact that the total mass of each transition measure μ_s in an LMP is at most one (cf. the proof of Proposition 7.1).

Definition 6.1 *Given a preordered ring R, $\mathcal{T}(R)$ is the preordered ring presented with generators the trace trees over R and Relations (3—9).*

Since the class of trace trees is closed under multiplication in $\mathcal{T}(R)$ it follows that a typical element of $\mathcal{T}(R)$ is equal to a linear combination (over $\mathbb{Z}$) of trace trees. In turn, this entails that prefixing by $a \in \mathrm{Act}$ extends uniquely to a selfmap of $\mathcal{T}(R)$ that distributes over finite sums, i.e., we write $a.0 = 0$ and $a.([t_1] + [t_2]) = [a.t_1] + [a.t_2]$.

Proposition 6.2 *If R is Archimedean then so is $\mathcal{T}(R)$.*

Proof. All the terms in the sum on the left-hand side of Inequality 9 are positive. This entails that each individual summand is dominated by the right-hand side, that is, $[\varepsilon.1_R] \sqsubseteq 1$ and $[a.1] \sqsubseteq 1$ for all $a \in \mathrm{Act}$. We use these inequalities and structural induction to verify that the Archimedean axiom holds for all trace trees.

For the base case, suppose $g \in R$. Since R is Archimedean, there exists $n \in \mathbb{N}$ such that $g \sqsubseteq n$ in R. Then $[\varepsilon.g] \sqsubseteq [\varepsilon.n] = n \cdot [\varepsilon.1_R] \sqsubseteq n$ (where the last inequality holds because $[\varepsilon.1_R] \sqsubseteq 1$).

The inductive case for prefixing by $a \in \mathrm{Act}$ is similar. Suppose t is a trace tree and $[t] \sqsubseteq n$. Then by monotonicity and linearity of prefixing in $\mathcal{T}(R)$ we have $[a.t] \sqsubseteq a.n = n \cdot [a.1] \sqsubseteq n$. The inductive case for product of trace trees is straightforward. This completes the proof that each trace tree is dominated by some $n \in \mathbb{N}$.

Finally, since each element of $\mathcal{T}(R)$ is equal to a linear combination of trace trees, the Archimedean axiom immediately follows. $\square$

Remark 6.3 *Given a preordered ring A, to define a monotone ring homomorphism $h : \mathcal{T}(R) \to A$ it suffices to define an interpretation in A of the trace trees over R that respects Relations (3—9). Note that Equations 5 and 6 force us to interpret multiplication of trace trees as multiplication in A, so we need only specify the value of h on trace trees of the form $a.t$ and $\varepsilon.r$.*

Next we complete the definition of the comonad $(\mathcal{T}, \xi, \delta)$. Note that in the sequel we omit square brackets when referring to trace trees as elements of $\mathcal{T}(R)$.

Definition 6.4 *The **comultiplication** $\delta : \mathcal{T} \Rightarrow \mathcal{T}^2$ has components $\delta_R : \mathcal{T}(R) \to \mathcal{T}^2(R)$ defined by the following clauses, where t is a trace tree over R and $r \in R$ (cf. Remark 6.3):*

$$\begin{aligned} \delta_R(a.t) &= a.\delta_R(t) + \varepsilon.(a.t) \\ \delta_R(\varepsilon.r) &= \varepsilon.\varepsilon.r\,. \end{aligned}$$

The counit $\xi : \mathcal{T} \Rightarrow Id$ has components $\xi_R : \mathcal{T}(R) \to R$ defined by

$$\begin{aligned} \xi_R(a.t) &= 0 \\ \xi_R(\varepsilon.r) &= r. \end{aligned}$$

Following Remark 6.3, it should be verified that the above definitions of δ_R and ξ_R respect Relations (3—9). This verification is routine: as a representative, we give details of the argument that δ_R respects Inequality 9.

$$\begin{aligned}
\delta_R(\varepsilon.1_R + \sum_{a\in\mathrm{Act}} a.1_{\mathcal{T}(R)}) &= \delta_R(\varepsilon.1_R) + \delta_R(\sum_{a\in\mathrm{Act}} a.1_{\mathcal{T}(R)}) \\
&= \varepsilon.\varepsilon.1_R + \sum_{a\in\mathrm{Act}} \delta_R(a.1_{\mathcal{T}(R)}) && [\text{Defn. of } \delta_R] \\
&= \varepsilon.\varepsilon.1_R + \sum_{a\in\mathrm{Act}} (a.1_{\mathcal{T}^2(R)} + \varepsilon.a.1_{\mathcal{T}(R)}) && [\text{Defn. of } \delta_R] \\
&= \varepsilon.(\varepsilon.1_R + \sum_{a\in\mathrm{Act}} a.1_{\mathcal{T}(R)}) + \sum_{a\in\mathrm{Act}} a.1_{\mathcal{T}^2(R)} && [\text{Eqn. } 4] \\
&\sqsubseteq \varepsilon.1_{\mathcal{T}(R)} + \sum_{a\in\mathrm{Act}} a.1_{\mathcal{T}^2(R)} && [\text{Eqn. } 9] \\
&\sqsubseteq 1_{\mathcal{T}^2(R)}\,. && [\text{Eqn. } 9]
\end{aligned}$$

Observe that comultiplication maps a trace tree t to the sum of all possible *decompositions* of t. First a simple example without branching: $\delta_R(a.b.c) = \varepsilon.(a.b.c) + a.\varepsilon.(b.c) + a.b.\varepsilon.c + a.b.c$. Next, an example with branching:

$$\delta_R(a.(b*c)) = \varepsilon.a.(b*c) + a.(\varepsilon.b*\varepsilon.c) + a.(\varepsilon.b*c) + a.(b*\varepsilon.c) + a.(b*c)\,.$$

Theorem 6.5 *$(\mathcal{T}, \delta, \xi)$ is a comonad on* **ORng**.

Proof. The counit laws are trivial. We will verify the associativity law for comultiplication. This asserts that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(R) \arrow[r, "\delta_R"] \arrow[d, "\delta_R"'] & \mathcal{T}^2(R) \arrow[d, "\mathcal{T}(\delta_R)"] \\
\mathcal{T}^2(R) \arrow[r, "\delta_{\mathcal{T}(R)}"'] & \mathcal{T}^3(R)
\end{tikzcd}$$

By Remark 6.3 it suffices to show that $\delta_{\mathcal{T}(R)}(\delta_R(t)) = \mathcal{T}(\delta_R)(\delta_R(t))$ for all trace trees t. We do this by structural induction on $t \in \mathcal{T}(R)$.

For the base case we observe that $\delta_{\mathcal{T}(R)}(\delta_R(\varepsilon.r)) = \varepsilon.\varepsilon.\varepsilon.r = \mathcal{T}(\delta_R)(\delta_R(\varepsilon.r))$ for all $r \in R$.

The inductive clause for prefixing is as follows:

$$\begin{aligned}
\delta_{\mathcal{T}(R)}(\delta_R(a.t)) &= \delta_{\mathcal{T}(R)}(a.\delta_R(t) + \varepsilon.a.t) && [\text{defn. of } \delta_R]\\
&= \delta_{\mathcal{T}(R)}(a.\delta_R(t)) + \delta_{\mathcal{T}(R)}(\varepsilon.a.t)\\
&= a.\delta_{\mathcal{T}(R)}(\delta_R(t)) + \varepsilon.a.\delta_R(t) + \delta_{\mathcal{T}(R)}(\varepsilon.a.t) && [\text{defn. of } \delta_{\mathcal{T}(R)}]\\
&= a.\delta_{\mathcal{T}(R)}(\delta_R(t)) + \varepsilon.a.\delta_R(t) + \varepsilon.\varepsilon.a.t && [\text{defn. of } \delta_{\mathcal{T}(R)}]\\
&= a.\mathcal{T}(\delta_R)(\delta_R(t)) + \varepsilon.a.\delta_R(t) + \varepsilon.\varepsilon.a.t && [\text{ind. hyp.}]\\
&= a.\mathcal{T}(\delta_R)(\delta_R(t)) + \varepsilon.(a.\delta_R(t) + \varepsilon.a.t) && [\text{Eqn. } 4]\\
&= a.\mathcal{T}(\delta_R)(\delta_R(t)) + \varepsilon.\delta_R(a.t) && [\text{defn. of } \delta_R]\\
&= \mathcal{T}(\delta_R)(a.\delta_R(t) + \varepsilon.a.t) && [\text{action of } \mathcal{T}(\delta_R)]\\
&= \mathcal{T}(\delta_R)(\delta_R(a.t))\,. && [\text{defn. of } \delta_R]
\end{aligned}$$

The inductive clause for multiplication straightforwardly follows from the fact that the components of δ, being ring maps, respect multiplication. ☐

7 Duality

The class of trace trees as originally defined in Section 5 can now be seen as the generators of $\mathcal{T}B(Y)$. Next we verify that the semantics of trace trees relative to an LMP $\mathcal{S} : X \to Y$, as given in Definition 5.1, uniquely specifies a map $\mathcal{T}B(Y) \to B(X+S)$ in **ORng**. To denote this map we reuse the notation $(-)_{\mathcal{S}}$ introduced in Definition 5.1.

Proposition 7.1 *Let $\mathcal{S} \colon X \to Y$ be an LMP, with $\mathcal{S} = (S, \mu)$. There is a unique monotone ring homomorphism*

$$\mathcal{T}B(Y) \xrightarrow{(-)_{\mathcal{S}}} B(X+S)$$

satisfying the following two clauses:

$$\begin{aligned}
(a.t)_{\mathcal{S}}(s) &= \int_S t_{\mathcal{S}}\, d\mu_{s,a}\\
(\varepsilon.g)_{\mathcal{S}}(s) &= \int_Y g\, d\mu_{s,\varepsilon}\,.
\end{aligned}$$

for all $g \in B(Y)$, trace trees $t \in \mathcal{T}B(Y)$, and $s \in X+S$.

Proof. By Remark 6.3 it suffices to verify that $(-)_{\mathcal{S}}$ respects Equations 3–9. Equations 3—6 are respected because integration is linear, and Inequalities 8 and 7 are respected because integration is monotone. It remains to verify that Inequality 9 is respected.

To this end, writing $t = 1_Y + \sum_{a \in \mathrm{Act}} a.1$ we have

$$
\begin{aligned}
t_{\mathcal{S}}(s) &= \int_Y d\mu_{s,\varepsilon} + \sum_{a\in \mathrm{Act}} \int_S d\mu_{s,a} \\
&= \mu_s(Y + \mathrm{Act}\times S) \\
&\leqslant 1 = t_{\mathcal{S}}(1)\,.
\end{aligned}
$$

□

We now come to the central definition of this paper: the dual of an LMP.

Definition 7.2 *Let $\mathcal{S} : X \to Y$ be an LMP and let $\pi_X : B(X+S) \to B(X)$ be given by $\pi_X(f) = f\,|_X$. Following on from Proposition 7.1, define $\widehat{\mathcal{S}} : \mathcal{T}B(Y) \to B(X)$ to be the following composition*

$$\mathcal{T}B(Y) \xrightarrow{(-)_{\mathcal{S}}} B(X+S) \xrightarrow{\pi_X} B(X)\,.$$

We call $\widehat{\mathcal{S}}$ the dual of $\mathcal{S}$. Notice that $\widehat{\mathcal{S}}$ is a map $B(Y) \to B(X)$ in the co-Kleisli category of the trace tree comonad $(\mathcal{T}, \xi, \delta)$. Later we will show that composition of LMPs, as defined in Section 3.1, corresponds to composition in the co-Kleisli category. However the remainder of this section is devoted to proving Theorem 5.2, which can now be reformulated as asserting that $\mathcal{S}$ is bisimilar to $\mathcal{S}'$ iff $\widehat{\mathcal{S}} = \widehat{\mathcal{S}'}$.

The proof of Theorem 5.2 involves completing $\mathcal{T}B(Y)$ to a C^*-algebra $\mathcal{A}(Y)$ and constructing a final LMP whose state space is the spectrum of $\mathcal{A}(Y)$. In this representation, a state of the final LMP is a character φ of $\mathcal{A}(Y)$. Such states have the following extensionality property: the value $\varphi(t)$ of φ on a given trace tree t is just the probability that φ, regarded as a state, can perform t.

7.1 A C^*-algebra of Trace Trees

Let **ARng** be the full subcategory of **ORng** consisting of Archimedean preordered rings. In this section we observe that **C*-Alg** is a reflective subcategory of **ARng**. Recall from Section 4 that an Archimedean preordered ring A is a C^*-algebra iff the additive group of A is torsion-free and divisible (equivalently, if A admits a $\mathbb{Q}$-algebra structure) and if A is complete in the norm (1).

Definition 7.3 *Given commutative rings A and B, the tensor product of A and B as Abelian groups can be turned into a ring by defining $(a\otimes b)\cdot(x\otimes y) = ax\otimes by$ and then extending linearly. This is the* ***ring tensor product*** *$A\otimes B$ of A and B.*

Note that the ring tensor product $\mathbb{Q}\otimes A$ is nothing but the free $\mathbb{Q}$-algebra over A. In case A is a preordered ring, we can equip $\mathbb{Q}\otimes A$ with the smallest preorder such that $0 \sqsubseteq q\otimes a$ whenever $0 \sqsubseteq q$ in $\mathbb{Q}$ and $0 \sqsubseteq a$. In this case it is clear that $\mathbb{Q}\otimes A$ inherits the Archimedean property from A.

Proposition 7.4 *The inclusion $U : \mathbf{C^*\text{-}Alg} \hookrightarrow \mathbf{ARng}$ has a left adjoint F.*

Proof. Write $\mathbf{ARng}_{\mathbb{Q}}$ for the subcategory of $\mathbf{ARng}$ consisting of the torsion-free divisible rings. We can factor U into two parts: the inclusion $U_1 : \mathbf{C^*\text{-}Alg} \hookrightarrow \mathbf{ARng}_{\mathbb{Q}}$ and the inclusion $U_2 : \mathbf{ARng}_{\mathbb{Q}} \hookrightarrow \mathbf{ARng}$. We show that both U_1 and U_2 have left adjoints. Indeed we have already observed that the map $A \mapsto \mathbb{Q} \otimes A$ gives a left adjoint to U_2. The left adjoint to U_1 is given by Cauchy completion, as we explain below.

A ring $A \in |\mathbf{ARng}_{\mathbb{Q}}|$ can be equipped with the seminorm (1) from Section 4. Let B denote the Cauchy completion of A in this norm, and write $\eta_1 : A \to B$ for the unit of the Cauchy-completion adjunction. Note that η_1 identifies all elements of A with zero norm, so it need not be injective. However, given $f \in A$, we will denote $\eta_1(f) \in B$ by just f.

We define a ring structure on B by $f+g = \lim_n(f_n+g_n)$ and $fg = \lim_n f_n g_n$, where $f, g \in B$ are such that $f = \lim_n f_n$ and $g = \lim_n g_n$ for $f_n, g_n \in A$. We also define a partial order on B by specifying the cone of positive elements. We say that $0 \sqsubseteq f$ if $f = \lim_n f_n$ for $f_n \in A$ with $0 \sqsubseteq f_n$. It is easy to show that the ring structure is well-defined, that B is a $\mathbb{Q}$-algebra, and that the order is Archimedean.

We can now consider two different norms on B: the norm it inherits as the Cauchy completion of A and the norm (1). It is straightforward that these two coincide, and we conclude that B is complete in the norm (1) and is therefore a C^*-algebra. $\square$

Definition 7.5 *Let $\mathcal{A}(Y)$ denote the reflection of $\mathcal{T}B(Y)$ in $\mathbf{C^*\text{-}Alg}$.*

Recall from Proposition 7.1 that an LMP $\mathcal{S} : X \to Y$ induces a monotone ring homomorphism $(-)_{\mathcal{S}} : \mathcal{T}B(Y) \to B(X+S)$. Since $B(X+S)$ is a C^*-algebra (cf. Example 4.2), by Proposition 7.4 the above map factors through $\mathcal{A}(Y)$ yielding a map (which we denote by the same name) $(-)_{\mathcal{S}} : \mathcal{A}(Y) \to B(X+S)$.

Write $\eta : Id \to UF$ for the unit of the adjunction defined in Proposition 7.4. The following proposition shows that $F(A)$ is free over A even if we consider maps that don't preserve multiplicative structure.

Proposition 7.6 *Let A be an Archimedean preordered ring, B a C^*-algebra, and $f : A \to UB$ a monotone function that is also a group homomorphism with respect to the additive structure of A and B. Then there is a unique $\mathbb{R}$-linear monotone map $\overline{f} : F(A) \to B$ such that $U\overline{f} \circ \eta = f$.*

Proof. The map $\overline{f}$ is defined exactly as if f were a monotone ring map: first f extends uniquely to a monotone $\mathbb{Q}$-linear map $\mathbb{Q} \otimes A \to B$ given by $q \otimes a \mapsto q \cdot f(a)$. This last map extends to an $\mathbb{R}$-linear map on the Cauchy completion of $\mathbb{Q} \otimes A$. $\square$

Note that prefixing by $a \in \mathrm{Act}$ is a monotone map $a.(-) : \mathcal{T}B(Y) \to \mathcal{T}B(Y)$ that is a homomorphism with respect to the additive group structure of $\mathcal{T}B(Y) \to \mathcal{T}B(Y)$. By Proposition 7.6 this extends to monotone $\mathbb{R}$-linear map $\mathcal{A}(Y) \to \mathcal{A}(Y)$.

7.2 A Universal LMP

We now define a universal LMP with state space $\mathrm{Spec}\,\mathcal{A}(Y)$ [4]. To manufacture the transition probabilities we use the Riesz representation theorem [21].

Theorem 7.7 (Riesz) *Let K be a compact Hausdorff space and $\varphi\colon C(K) \to \mathbb{R}$ a monotone $\mathbb{R}$-linear map. Then there is a unique positive Borel measure μ on K such that $\varphi(f) = \int f d\mu$ for all $f \in C(K)$. The total mass of μ is given by $\varphi(1)$.*

Given $\varphi \in \mathrm{Spec}\,\mathcal{A}(Y)$, define its *derivative* $\varphi_a : \mathcal{A}(Y) \to \mathbb{R}$ with respect to $a \in \mathrm{Act}$ by $\varphi_a(f) = \varphi(a.f)$. Then φ_a is monotone and linear in the sense of Theorem 7.7 since both φ and the prefixing map $a.(-)$ are monotone and linear on $\mathcal{A}(Y)$.

Definition 7.8 *Define $\mu : \mathrm{Spec}\,\mathcal{A}(Y) \longrightarrow \mathcal{M}(Y + (\mathrm{Act} \times \mathrm{Spec}\,\mathcal{A}(Y))$ as follows.*

Given $\varphi \in \mathrm{Spec}\,\mathcal{A}(Y)$ and $a \in \mathrm{Act}$, define $\mu_{\varphi,a}$ to be the Borel measure on $\mathrm{Spec}\,\mathcal{A}(Y)$ corresponding by Theorem 7.7 to the linear map

$$C(\mathrm{Spec}\,\mathcal{A}(Y)) \cong \mathcal{A}(Y) \xrightarrow{\varphi_a} \mathbb{R}\,.$$

(Note that the isomorphism $C(\mathrm{Spec}\,\mathcal{A}(Y)) \cong \mathcal{A}(Y)$ comes from Theorem 4.4.) Furthermore, define a positive Borel measure $\mu_{\varphi,\varepsilon}$ on Y by $\mu_{\varphi,\varepsilon}(A) = \varphi(\varepsilon.\chi_A)$ for each measurable $A \subseteq Y$. This completes the definition of μ_φ and it remains to observe that μ_φ is a subprobability measure since its total mass is given by

$$\begin{aligned}
\mu_{\varphi,\varepsilon}(Y) + \textstyle\sum_{a\in\mathrm{Act}} \mu_{\varphi,a}(\mathrm{Spec}\,\mathcal{A}(Y)) &= \varphi(\varepsilon.1_Y) + \textstyle\sum_{a\in\mathrm{Act}} \varphi_a(1) \\
&= \varphi(\varepsilon.1_Y + \textstyle\sum_{a\in\mathrm{Act}} a.1) \\
&\leqslant \varphi(1) && \textit{[Eqn. 9]} \\
&= 1\,.
\end{aligned}$$

Definition 7.8 specifies an LMP of type $\emptyset \to Y$. The following proposition formalises the idea that this is a universal LMP on the space of exit points Y. It says that for an arbitrary LMP $\mathcal{S} : X \to Y$ we can augment the universal LMP by specifying a space of entry points X, thus obtaining an LMP $\mathcal{S}_* : X \to Y$, such that there is a zig-zag map from $\mathcal{S}$ to $\mathcal{S}_*$.

[4] By definition of $\mathcal{A}(Y)$ there is a bijection between $\mathrm{Spec}\,\mathcal{A}(Y)$ and $\mathbf{ORng}(\mathcal{T}B(Y), \mathbb{R})$. Nevertheless it is convenient to work with $\mathcal{A}(Y)$ since there is no way to recover $\mathcal{T}B(Y)$ from $\mathbf{ORng}(\mathcal{T}B(Y), \mathbb{R})$.

Definition 7.9 *Given an LMP $\mathcal{S} : X \to Y$, define $\pi : X \to \operatorname{Spec} \mathcal{A}(Y)$ by $\pi(x)(f) = f_{\mathcal{S}}(x)$. Furthermore write $\mathcal{S}_* : X \to Y$ for the LMP with state space $\operatorname{Spec} \mathcal{A}(Y)$ and transition map*

$$[\mu \circ \pi, \mu] : X + \operatorname{Spec} \mathcal{A}(Y) \longrightarrow \mathcal{M}(Y + (\mathrm{Act} \times \operatorname{Spec} \mathcal{A}(Y)) ,$$

where μ is as in Definition 7.8.

Proposition 7.10 *The function $h\colon S \to \operatorname{Spec} \mathcal{A}(Y)$ defined by $h(s)(f) = f_{\mathcal{S}}(s)$ is a zig-zag map $\mathcal{S} \to \mathcal{S}_*$.*

Proof. Let $\rho : X + S \to \mathcal{M}(Y + (\mathrm{Act} \times S))$ be the transition function of $\mathcal{S}$. According to Definition 2.3, $h\colon S \to \operatorname{Spec} \mathcal{A}(Y)$ is a zig-zag map iff (i) $\rho_{s,a} \circ h^{-1}$ and $\mu_{h(s),a}$ are identical measures on $\operatorname{Spec} \mathcal{A}(Y)$ for each $s \in S$ and $a \in \mathrm{Act}$, and (ii) $\rho_{s,\varepsilon}$ and $\mu_{h(s),\varepsilon}$ are identical measures on Y for each $s \in S$. We will demonstrate that (i) holds in this case; the justification of (ii) is similar.

Given $f \in \mathcal{A}(Y)$ let $\widehat{f} \in C(\operatorname{Spec} \mathcal{A}(Y))$ be defined by $\widehat{f}(\varphi) = \varphi(f)$. Note that $\widehat{f}(h(s)) = h(s)(f) = f_{\mathcal{S}}(s)$ for all $s \in S$. Thus we have

$$\begin{aligned}
\int \widehat{f}\, d(\rho_{s,a} \circ h^{-1}) &= \int (\widehat{f} \circ h)\, d\rho_{s,a} \\
&= \int f_{\mathcal{S}}\, d\rho_{s,a} \\
&= (a.f)_{\mathcal{S}}(s) && [\text{defn. of } (-)_{\mathcal{S}}] \\
&= h(s)(a.f) \\
&= \int \widehat{f}\, d\mu_{h(s),a} \,. && [\text{by Defn. 7.8}]
\end{aligned}$$

By the Riesz representation theorem, two Borel measures on $\operatorname{Spec} \mathcal{A}(Y)$ are equal iff their respective integrals against any continuous function are equal. But each continuous function on $\operatorname{Spec} \mathcal{A}(Y)$ has the form $\widehat{f}$ for some $f \in \mathcal{A}(Y)$. We conclude that $\rho_{s,a} \circ h^{-1} = \mu_{h(s),a}$. □

We obtain the following corollary, which is a restatement of Theorem 5.2: if two LMPs have the same dual then they are bisimilar.

Corollary 7.11 *LMPs $\mathcal{S}, \mathcal{S}' : X \to Y$ are bisimilar if $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}'$.*

Proof. According to Definition 7.9, if $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}'$ then $\mathcal{S}_* = \mathcal{S}'_*$. But then two applications of Proposition 7.10 yield a cospan $\mathcal{S} \longrightarrow \mathcal{S}_* = \mathcal{S}'_* \longleftarrow \mathcal{S}'$ of zig-zag maps, showing that $\mathcal{S}$ and $\mathcal{S}'$ are bisimilar according to Definition 2.4. □

8 Structure of the Dual Category

In this section we characterise the dual category of **LMP**, which we call **Eval**.

Definition 8.1 *The objects of the category* **Eval** *are the measurable spaces, and an arrow $X \to Y$ is a homomorphism $\mathcal{T}B(X) \to B(Y)$ of preordered rings. Composition in **Eval** is just as in the co-Kleisli category of $\mathcal{T}$. We call the morphisms in this category **evaluations**.*

In this section we extensively rely on Remark 6.3, that is, we define a monotone ring map $h : \mathcal{T}B(Y) \to B(X)$ just by specifying the values $h(\varepsilon.g)$ and $h(a.t)$ for each $g \in B(Y), a \in \mathrm{Act}$ and trace tree t. This suffices to define h on the set of all trace trees over $B(Y)$, and it then remains to check that h respects the relations in the presentation of $\mathcal{T}B(Y)$.

Example 8.2 *Recall from Section 3 that binary coproducts in **LMP** are given by the stochastic relations $inl : X \to X+Y$ and $inr : Y \to X+Y$. Here we describe the dual maps $\widehat{inl} = \pi_1 : \mathcal{T}B(X+Y) \to B(X)$ and $\widehat{inr} = \pi_2 : \mathcal{T}B(X+Y) \to B(Y)$. These are defined by*

$$\begin{aligned} \pi_1(a.t) &= 0 \\ \pi_1(\varepsilon.g) &= g \mid_X , \end{aligned}$$

and

$$\begin{aligned} \pi_2(a.t) &= 0 \\ \pi_2(\varepsilon.g) &= g \mid_Y . \end{aligned}$$

The fact that the coproduct injections are stateless corresponds to the fact that π_1 and π_2 map any trace tree not of the form $\varepsilon.g$ to 0. The subcategory of maps with this property is (isomorphic to) **SPT**, the category of stochastic predicate transformers of Definition 3.2.

The following proposition shows that composition of LMPs corresponds to composition in **Eval**.

Proposition 8.3 *Given LMPs $\mathcal{S}_1 \colon X \to Y$ and $\mathcal{S}_2 \colon Y \to Z$, $(\mathcal{S}_2 \circ \mathcal{S}_1)\widehat{} = \widehat{\mathcal{S}_1} \circ \widehat{\mathcal{S}_2}$.*

Proof. Write $\mathcal{S}_1 = (S, \mu)$, $\mathcal{S}_2 = (S', \mu')$ and, following Section 3.1, denote the transition function of $\mathcal{S}_2 \circ \mathcal{S}_1$ by ρ.

We show that the following two statements hold for all trace trees $t \in \mathcal{T}B(Z)$.

(i) $t_{\mathcal{S}_2 \circ \mathcal{S}_1}(s) = t_{\mathcal{S}_2}(s)$ for all $s \in S'$.

(ii) $t_{\mathcal{S}_2 \circ \mathcal{S}_1}(s) = ((\mathcal{T}\widehat{\mathcal{S}_2} \circ \delta_{B(Z)})(t))_{\mathcal{S}_1}(s)$ for all $s \in S + X$.

Before proving them, we observe that (ii) yields our desired conclusion. Indeed for $t \in \mathcal{T}B(Z)$ and $x \in X$ we have

$$
\begin{aligned}
(\mathcal{S}_2 \circ \mathcal{S}_1)\widehat{\ }(t)(x) &= t_{\mathcal{S}_2 \circ \mathcal{S}_1}(x) && \text{[Defn. 7.2]} \\
&= ((\mathcal{T}\widehat{\mathcal{S}_2} \circ \delta_{B(Z)})(t))_{\mathcal{S}_1}(x) && \text{[by (ii)]} \\
&= (\widehat{\mathcal{S}}_1 \circ \mathcal{T}\widehat{\mathcal{S}}_2 \circ \delta)(t)(x) && \text{[Defn. 7.2]} \\
&= (\widehat{\mathcal{S}_1} \circ \widehat{\mathcal{S}_2})(t)(x) && \text{[co-Kleisli composition]}
\end{aligned}
$$

It remains to prove (i) and (ii). Statement (i) says that the probability of performing a trace tree starting from $s \in S'$ does not depend on whether we regard s as a state of $\mathcal{S}_2$ or of $\mathcal{S}_2 \circ \mathcal{S}_1$. The proof is straightforward given the fact that for $s \in S'$ and $E \subseteq Z + (\mathrm{Act} \times S')$ we have $\mu'_s(E) = \rho_s(E)$.

We prove (ii) by structural induction on trace trees.

$$
\begin{aligned}
(a.t)_{\mathcal{S}_2 \circ \mathcal{S}_1}(s) &= \int_{S+S'} t_{\mathcal{S}_2 \circ \mathcal{S}_1}\, d\rho_{s,a} \\
&= \int_S t_{\mathcal{S}_2 \circ \mathcal{S}_1}\, d\mu_{s,a} + \int_Y \lambda y. \left(\int_{S'} t_{\mathcal{S}_2 \circ \mathcal{S}_1}\, d\mu'_{y,a} \right) d\mu_{s,\varepsilon} && \text{[defn. of } \rho_{s.a}] \\
&= \int_S t_{\mathcal{S}_2 \circ \mathcal{S}_1}\, d\mu_{s,a} + \int_Y \lambda y. \left(\int_{S'} t_{\mathcal{S}_2}\, d\mu'_{y,a} \right) d\mu_{s,\varepsilon} && \text{[by (i)]} \\
&= \int_S t_{\mathcal{S}_2 \circ \mathcal{S}_1}\, d\mu_{s,a} + \int_Y (a.t)_{\mathcal{S}_2}\, d\mu_{s,\varepsilon} && \text{[Defn. 5.1]} \\
&= \int_S ((\mathcal{T}\widehat{\mathcal{S}_2} \circ \delta_{B(Z)})(t))_{\mathcal{S}_1}\, d\mu_{s,a} + \int_Y (a.t)_{\mathcal{S}_2}\, d\mu_{s,\varepsilon} && \text{[ind. hyp. (ii)]} \\
&= (a.\mathcal{T}\widehat{\mathcal{S}_2}(\delta_{B(Z)}(t)))_{\mathcal{S}_1}(s) + (\varepsilon.\widehat{\mathcal{S}_2}(a.t))_{\mathcal{S}_1}(s) && \text{[Defn. 5.1]} \\
&= (a.\mathcal{T}\widehat{\mathcal{S}_2}(\delta_{B(Z)}(t)) + \varepsilon.\widehat{\mathcal{S}_2}(a.t))_{\mathcal{S}_1}(s) \\
&= (\mathcal{T}\widehat{\mathcal{S}_2}(a.\delta_{B(Z)}(t) + \varepsilon.a.t))_{\mathcal{S}_1}(s) && \text{[action of } \mathcal{T}\widehat{\mathcal{S}_2}] \\
&= \mathcal{T}\widehat{\mathcal{S}_2}(\delta_{B(Z)}(a.t))_{\mathcal{S}_1}(s) && \text{[Defn. 6.4]}
\end{aligned}
$$

□

Corollary 8.4 *Composition in **LMP** is associative.*

Proof. This follows immediately from the fact that composition in the co-Kleisli category of $\mathcal{T}$ is associative. □

9 Conclusions

This paper characterised bisimulation equivalence of LMPs as trace-tree equivalence. This characterisation was proved using Stone duality for real C^*-

algebras to construct a universal LMP as the spectrum of a C^*-algebra of trace trees. The fact that bisimilarity has such a simple characterisation as a trace-like equivalence corresponds to the intuition that probabilistic branching is better behaved than genuine nondeterminism.

We also considered LMPs with distinguished sets of entries and exits as generalised stochastic relations. Using the notion of trace tree over a ring, we defined a comonad on **ORng** and established a duality between LMPs and maps in the co-Kleisli category of the comonad.

One aspect of the category **LMP** that we have not touched on is its partially additive structure. In fact, it is straightforward to generalise the partially additive structure of **SRel** (as outlined in [20]) to **LMP**, and thus to define an iteration operation on **LMP**. A question for future work is to isolate some extra structure on the trace-tree comonad $\mathcal{T}$ that corresponds to iteration of LMPs, just as comultiplication corresponds to composition of LMPs. Here we are thinking of a decomposition of trace trees, along the lines of Definition 6.4, that captures the sum-over-paths intuition that lies behind the definition of iteration in **LMP**.

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