Semantics of first order parametric specifications

Duško Pavlović

Kestrel Institute, Palo Alto, USA
dusko@kestrel.edu

Abstract. Parametricity is one of the most effective ways to achieve compositionality and reuse in software development. Parametric specifications have been thoroughly analyzed in the algebraic setting and are by now a standard part of most software development toolkits. However, an effort towards classifying, specifying and refining algorithmic theories, rather than mere datatypes, quickly leads beyond the realm of algebra, and often to full first order theories. We extend the standard semantics of parametric specifications to this more general setting.
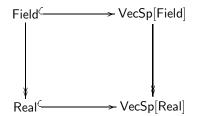
The familiar semantic characterization of parametricity in the algebraic case is expressed in terms of the free functor, i.e. using the initial models. In the general case, initial models may not exist, and the free functor is not available. Various syntactic, semantic, and abstract definitions of parametricity have been offered, but their exact relationships are often unclear. Using the methods of categorical model theory, we establish the equivalence of two well known, yet so far unrelated, definitions of parametricity, one syntactic, one semantic. Besides providing support for both underlying views, and a way for aligning the systems based on each of them, the offered general analysis and its formalism open several avenues for future research and applications.

1 Introduction

1.1 Parametric specifications

The idea of *parametric polymorphism* goes back to Strachey [26] and refers to code reusable over any type that may be passed to it as a parameter. If a type is viewed as a set of logical invariants of the data, this idea naturally extends to the software specifications, as the logical theories capturing requirements and allowing their refinement. The idea of parametric specifications was proposed early on and became a standard part of specification theory (cf. e.g. [8, 12, 13] and the references therein).

A standard nontrivial example of a parametric specification is a presentation of the theory of vector spaces, with the theory of fields as its parameter. The idea is that refining the parameter, in this case the subtheory referring to scalars, yields a consistent refinement of the larger theory, usually called the *body*. Formally, we are given a theory VecSp and a distinguished subtheory Field ↪ VecSp. The refinement is realized by the pushout in the category of specifications [4,

11].

$$
\begin{array}{ccc}
\mathsf{Field} & \hookrightarrow & \mathsf{VecSp[Field]} \\
\downarrow & & \downarrow \\
\mathsf{Real} & \hookrightarrow & \mathsf{VecSp[Real]}
\end{array}
$$

The functoriality of the pushout operation ensures the compositionality of the refinements.

Of course, not every interpretation of one specification in another allows this. For instance, if instead of Field, just the theory of rings is taken as the parameter of VecSp, some consistent refinements of the parameter will induce inconsistencies in the body. Some models of the parameter therefore do not correspond to models of the body.

Some syntactic parametricity conditions, ensuring that consistent refinements of the parameter induce consistent refinements of the body, were proposed early on [9, 14]. However, the analogous semantic characterizations, ensuring that models of the parameter induce models of the body, were given only in terms of free functors, which only exist for (essentially) algebraic specifications, i.e. those stated using just operations and equations (and simple implications between them). In [9], cofree functors were analyzed as well, but for a general first order theory, they may not exist either. E.g., the theories of fields, Hilbert spaces, or linear orders do not have either intial or final models.

Algebraic specifications do suffice for great many practical tasks and offer a fruitful ground for theory [8]. However, when it comes to systems for code synthesis, like SPECWARE$^{\mathrm{TM}}$ [28], where it is essential to compositionally refine and implement not only abstract datatypes, but also abstract algorithmic theories, algebraic specifications become increasingly insufficient, and initial and final semantics do not apply.

On one hand, a syntactic form of parametricity for general specifications has been used in practice and in the literature [12, 13]. On the other hand, in [6], a semantical definition of parametricity was proposed, independent of the existence of initial or final models. However, it seems that neither the semantic characterization of the former nor the syntactic characterization of the latter have been worked out. Abstracting away from the concrete meaning of parametricity, some interesting structures have been built, applicable to parametric specifications in general [5, 25], yet no statement tieing together the syntactic and the semantic intuitions seems to have been proved. The purpose of the present paper is to try to bridge this gap, while providing some evidence of the applicability of *categorical model theory* to the study of general software specifications.[1]

---

[1] In contrast, the purported algebraicizing of general specifications in higher order logic by presenting the first order theorems as higher order equations only shifts the

1.2 Elements of categorical model theory

The functorial semantics of algebraic theories goes back to the sixties, to Lawvere's thesis [16]. The theory of categorical universal algebra which arose from it is summarized in [22]. An important step beyond algebra is the study of locally presentable categories [10], which come about as the model categories of limit theories, a wider, yet essentially restricted class. The full scope of first order logic was covered by categorical model theory rather slowly, throughout the seventies and eighties, as some parts tend to be technically rather demanding. Good accounts of the more accessible parts are [1, 20, 21].

The main idea of functorial semantics is to

- present logical theories as *classifying* categories with structure, so as to
- obtain their models as structure preserving functors to Set, with homomorphisms between them as natural transformations.

The resulting categories of models will always be *accessible*, i.e. have directed colimits and a suitable generating set. Conversely, every accessible category can be obtained as the category of models of a first order theory, possibly infinitary. Categorical model theory is thus the study of accessible categories and the way they arise from theories. There is a very general Stone-type duality between the first order theories, presented as categories, and the induced categories of models [19], but it is quite involved in technical details, and it is not clear whether it can be brought into a practically useful form.

But without going into the formal duality, one can still systematically explore the relationships between the *syntactic* and the *semantic* aspects of theories, by analyzing functors between their categorical presentations. In particular, for any two first order theories $\mathbb{A}$ and $\mathbb{B}$, presented as classifying categories, one can align the properties of the logical interpretations, which can be captured as functors $F : \mathbb{A} \longrightarrow \mathbb{B}$, and the induced forgetful, or "reduct", functors $F^{\#} : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ between the corresponding categories of models.

This is a typical task for semantics of software specifications: analyze how a particular class of syntactical manipulations with theories is reflected on their models, and on the computations that may be built on top. We shall show that a syntactic definition of parametric specification, viewed as a property of the interpretation functor $F : \mathbb{A} \longrightarrow \mathbb{B}$, is equivalent to an independent semantic definition, stated in terms of the "reduct" functor $F^{\#} : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$.

1.3 Outline of the paper

In the next section, we describe the concrete constructions of classifying categories, explain how interpretations are captured as functors between them, and how the idea of parametricity fits into this setting.

In section 3 we list some abstract preliminary results that align the syntactic and semantic properties of functors.

---

problems from the large but familiar area of first order model theory to the scarcely cultivated field of higher order algebra.

Finally, in section 4, we derive the main result: the equivalence of a syntactic form of parametricity, in the spirit of [12, 13], and a semantic form, as in [6], both adapted only to a common categorical setting.

2 Theories and models, categorically

2.1 Classifying categories

The simplest classifying category is the *Lawvere clone* $\mathbb{C}_{\mathcal{T}}$ of an algebraic theory $\mathcal{T}$, say single sorted. Its objects can be viewed as natural numbers (*viz* the arities), while a morphism from m to n is an n-tuple of the elements of the free algebra in m generators, i.e. a function $n \longrightarrow Tm$, where T denotes the free algebra constructor.[2] A crucial observation from Lawvere's thesis [16] is that $\mathbb{C}_{\mathcal{T}}$ classifies $\mathcal{T}$-algebras, in the sense that they exactly correspond to the product preserving functors $\mathbb{C}_{\mathcal{T}} \longrightarrow \mathsf{Set}$, while the $\mathcal{T}$-homomorphisms correspond to the natural transformations between them. Indeed, since n in $\mathbb{C}_{\mathcal{T}}$ appears as the product of n copies of 1, the product preservation ensures that the functors $\mathbb{C}_{\mathcal{T}} \longrightarrow \mathsf{Set}$ trace the operations with the correct arities. The equations of $\mathcal{T}$ are then enforced by the functoriality. Detailed explanations of the functorial semantics of algebraic theories can be found e.g. in [22].

If models of more general theories are to be captured as functors, some additional preservation properties will be needed, in order to enforce satisfaction of not necessarily equational formulas, that may express more than mere commutativity conditions. There are several well known frameworks for building suitable classifying categories and developing functorial semantics for general first order theories, the most "categorical" being probably sketches [3, 20]. We shall however work in the setting of *coherent categories* [21], closest to the original geometric spirit of categorical logic, because they seem to allow the quickest and perhaps the most intuitive approach to the matters presently of interest.

2.2 Coherent categories

Let $\mathcal{T}$ be a multisorted first order theory with equality. For simplicity, we assume that it is purely relational: operations are captured by their graphs. Moreover, $\mathcal{T}$ is assumed to be generated by a set of axioms in *coherent* logic, i.e. using finitary $\wedge$ and $\vee$, including the empty ones, $\top$ and $\bot$, and the quantifier $\exists$. The underlying logic can be classical or intuitionistic. We cannot go into the details here, but reducing finitary first order logic to its coherent fragment is a fairly standard technical device (see [1, 2, 21] and especially the informative introduction of [20]). The extension to infinitary logic is justified by stable and natural categories of models and is routinely handled by extending the classifying constructions. However, some of the proofs presented below essentially depend on the finiteness assumption.

---

[2] So if $\mathcal{T}$ is presented by the monad T, the classifier $\mathbb{C}_{\mathcal{T}}$ is the dual of the induced Kleisli category, restricted to natural numbers.

Formally, the theory $\mathcal{T}$ can be viewed as a preorder: the underlying set $|\mathcal{T}|$ of well-formed formulas is generated by its language, while the entailment preorder $\vdash$ is generated by its axioms. The rough idea is to capture the well-formed formulas of $\mathcal{T}$ as the objects of the classifying category $\mathbb{C}_{\mathcal{T}}$, and the theorems of $\mathcal{T}$ as the morphisms of $\mathbb{C}_{\mathcal{T}}$.

The passage from the formulas of $\mathcal{T}$ to the objects of $\mathbb{C}_{\mathcal{T}}$ requires an adjustment: the formulas must be viewed modulo variable renaming, i.e. α-conversion $\phi(x) \sim \phi(y)$, where x and y are vectors of variables. Note that this is not a congruence with respect to the logical operations, because e.g. $\phi(x) \wedge \phi(y) \not\sim \phi(x) \wedge \phi(x)$.

The passage from theorems of $\mathcal{T}$ to morphisms of $\mathbb{C}_{\mathcal{T}}$ requires a similar adjustment: modulo the logical equivalence $\varphi \dashv\vdash \psi$, which means that $\varphi \vdash \psi$ and $\psi \vdash \varphi$. The definition is thus

$$
|\mathbb{C}_{\mathcal{T}}| = |\mathcal{T}|/ \sim
$$
$$
\mathbb{C}_{\mathcal{T}}\left(\alpha(x), \beta(y)\right) = \big\{\vartheta(x,y) \in \mathcal{T} \mid \vartheta(x,y) \vdash \alpha(x) \wedge \beta(y),
$$
$$
\alpha(x) \vdash \exists y.\ \vartheta(x,y),
$$
$$
\vartheta(x,y') \wedge \vartheta(x,y'') \vdash y' \equiv y''\big\}/ \dashv\vdash
$$

where x and y are disjoint strings of variables, always available by renaming[3], and $\equiv$ is the equality predicate in $\mathcal{T}$. The identities in $\mathbb{C}_{\mathcal{T}}$ are induced by the equality predicates, and the composition of $\vartheta(x,y)$ and $\varrho(y,z)$ is $\exists y.\ \vartheta(x,y) \wedge \varrho(y,z)$.

The logical structure of $\mathcal{T}$ induces the categorical structure of $\mathbb{C}_{\mathcal{T}}$:

- *finite limits* are constructed using the conjunction and the variable tupling, starting from the true predicates $\top(x)$ over each sort;
- *regular epi-mono factorisations* are constructed using the existential quantifier; and finally
- *joins of the subobjects* correspond to the disjunctions.

These three structural components constitute a coherent category and are preserved by coherent functors. Theories in coherent logic generate coherent classifying categories; conversely, each small coherent category classifies a coherent theory. Coherent functors preserve the truth of the theorems in coherent logic. The reader may wish to work out the details of this correspondence or to consult some of the mentioned references.

A reader familiar with the functorial semantics of algebra has perhaps already noticed that the coherent classifier of an algebraic theory contains the corresponding Lawvere clone as a full subcategory, namely the one spanned by the true formulas $\top(x)$, one for each arity x. Indeed, the coherent classifier of an algebraic theory is the coherent completion of its Lawvere clone. The coherent classifiers have a richer set of objects, in order to impose the preservation of more general axioms; but simpler theories can be captured by smaller classifiers.

---

[3] By abuse of notation, $\alpha(x)$, $\beta(y)$ and $\vartheta(x,y)$ denote their equivalence classes $[\alpha]$, $[\beta]$ and $[\vartheta]$ modulo $\sim$.

2.3 Interpretations and models

The upshot of coherent classifying categories is thus that the coherent functors, preserving the coherent structure, preserve the coherent logic, and thus enforce the satisfaction of the coherent theorems, represented as the morphisms in coherent categories. A coherent functor $\mathbb{C}_{\mathcal{T}} \longrightarrow \mathbb{C}_{\mathcal{U}}$ can thus be viewed as a sound interpretation of the theory $\mathcal{T}$ in the theory $\mathcal{U}$. But since every small coherent category $\mathbb{A}$ can be obtained as the classifier $\mathbb{C}_{\mathcal{T}}$ of some coherent theory $\mathcal{T}$, every coherent functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ can be understood logically, as such an interpretation.

Although it is not small, Set has all the coherent structure, and the coherent functors $\mathbb{C}_{\mathcal{T}} \longrightarrow \mathsf{Set}$ are exactly the models of $\mathcal{T}$. The natural transformations are the $\mathcal{T}$-homomorphisms, preserving all the definable operations. For every small coherent $\mathbb{A}$, we shall denote by $\mathsf{Mod}(\mathbb{A})$ the category of coherent functors $\mathbb{A} \longrightarrow \mathsf{Set}$. This is the category of models of $\mathbb{A}$. As pointed out before, categories of the form $\mathsf{Mod}(\mathbb{A})$ are accessible, and by allowing infinite disjunctions, one could get (an equivalent version of) every accessible category in this form [1, ch. 5].

On the other hand, by precomposition, every coherent functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ induces a *"reduct"* $F^{\#} : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$, reinterpreting a model $N : \mathbb{B} \longrightarrow \mathsf{Set}$ of $\mathbb{B}$ as a model $NF : \mathbb{A} \longrightarrow \mathsf{Set}$ of $\mathbb{A}$. This is the arrow part of the Mod-construction, which yields an indexed category $\mathsf{Mod} : \mathsf{Coh}^{op} \longrightarrow \mathsf{CAT}$, where Coh is the category of small coherent categories and functors, and CAT is the metacategory of categories. Mod thus assigns a semantics to each coherent theory $\mathcal{T}$, classified by a coherent category $\mathbb{C}_{\mathcal{T}}$; in other words, it maps each theory $\mathcal{T}$ to its category of models, captured as coherent functors $\mathbb{C}_{\mathcal{T}} \longrightarrow \mathsf{Set}$.

The semantical functor Mod is an instance of a *specification frame* in the sense of Ehrig and Große-Rhode [6]. Specification frames are indexed categories, construed as some abstract model category assignments, like Mod. In these terms, Ehrig and Große-Rhode proposed a semantical definition of parametric specifications, which will be analyzed in the sequel.

2.4 Parametrized specifications as functors:
syntactic vs semantic definitions

A reader unfamiliar with coherent logic may wish to write down, as a quick exercise, say, the coherent theories of fields and vector spaces and analyze their classifying categories. The classifying category Field is of course a subcategory of the classifying category VecSp. The obvious functor $\mathsf{Field} \hookrightarrow \mathsf{VecSp}$ is full and faithful. This means that the theory of vector spaces is conservative over the theory of fields: no new theorems about the scalars can be proved using the vectors. Moreover, $\mathsf{Field} \hookrightarrow \mathsf{VecSp}$ is also a powerful functor: each subobject of an object in the image is also in the image. This means that every predicate on scalars, expressible in the theory of vector spaces, is already expressible in the theory of fields.

The embedding $\mathsf{Field} \hookrightarrow \mathsf{VecSp}$ is a typical parametric specification, defined syntactically, as in [12, 13]. Viewed in the setting of classifying categories, a parametric specification is thus a coherent functor $F : \mathbb{A} \longrightarrow \mathbb{B}$, which is full, faithful and powerful.

On the semantic side, as already mentioned, Ehrig and Große-Rhode [6] have proposed an abstract definition of parametricity, applicable to the functor $\mathsf{Mod} : \mathsf{Coh}^{op} \longrightarrow \mathsf{CAT}$. Omitting the presentation details, a parametric specification is, according to this definition, an interpretation $F : \mathbb{A} \longrightarrow \mathbb{B}$, such that the induced functor $F^{\#} : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ is a retraction, i.e. there is a functor $\Phi : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$ with $F^{\#} \circ \Phi \cong \mathrm{Id}$. In words, Φ maps each model M of the parameter $\mathbb{A}$ into a model $N = \Phi M$ of the body $\mathbb{B}$ in such a way that the forgetful functor $F^{\#}$ restores an isomorphic copy[4] of M. Such a functor Φ, which nondestructively expands a model, is said to be *persistent* [8, sec. 10B].

In the present paper, we shall show that the above two definitions are roughly equivalent: a coherent functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is full, faithful and powerful if and only if $F^{\#} : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ is a retraction, in the strong sense that every splitting of its object part can be refined, by taking quotients, into a splitting functor.

Completeness view. When an indexed family of sets $\{B_x | x \in A\}$ is represented as a function $f : B \longrightarrow A$, with $B_x = f^{-1}(x)$, an indexed element $b \in \prod_{x \in A} B_x$ becomes a splitting $\phi : A \longrightarrow B$, $f \circ \phi = \mathrm{id}$, with $b_x = \phi(x) \in B_x$.

Similarly, a specification $\mathbb{B}$ parametrized over $\mathbb{A}$ can be thought of as a family of the instances of $\mathbb{B}$ indexed over the instances of $\mathbb{A}$. In particular, the functor $F^{\#} : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ can be construed as a family of $\mathbb{B}$-models indexed over $\mathbb{A}$-models. A splitting $\Phi : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$, $F^{\#} \circ \Phi \cong \mathrm{Id}$, then becomes an indexed model of $\mathbb{B}$, parametrized over $\mathbb{A}$.

According to this view, a persistent functor is thus an indexed model. The parametricity of theories lifts to the parametricity of their models: the semantical definition of parametric specification, described above, boils down to the requirement that there is a parametric model of the body indexed over the models of the parameter.

The equivalence of the semantic and the syntactic definitions of parametricity, which we are about to establish, thus becomes a soundness-and-completeness theorem, in indexed form.

3 Syntactic vs semantic properties of functors

3.1 Preliminaries

We begin by listing some useful terminology and facts from the general functorial calculus.

---

[4] The original definition actually requires that M is recovered on the nose, i.e. that the strict equality $F^{\#} \circ \Phi = \mathrm{Id}$ holds. But in abstract functorial calculus, this is almost never possible.

Definition 1. *A functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is said to be*

embedding: *if it is full and faithful;*
subcovering: *if for every object $B \in \mathbb{B}$ there is a finite diagram D in $\mathbb{B}$, such that (1) B is the colimit of D, and (2) for every node D_i of D there is some A_i in $\mathbb{A}$ and a monic $D_i \rightarrowtail FA_i$ in $\mathbb{B}$;*
subobject covering: *if every $B \in \mathbb{B}$ is a subobject of some $FA, A \in \mathbb{A}$ (in other words, if it is subcovering and the diagrams D can be chosen to have one node and no edges);*
powerful: *if all subobjects[5] of FA in $\mathbb{B}$ lie in the image of F. More precisely, for every monic $D \overset{d}{\rightarrowtail} FA$ in $\mathbb{B}$ there is a monic $S \overset{s}{\rightarrowtail} A$ in $\mathbb{A}$ and an isomorphism $i : D \overset{\sim}{\longrightarrow} FS$ such that $d = Fs \circ i$;*
retraction: *if it has a right inverse (i.e., there is $G : \mathbb{B} \longrightarrow \mathbb{A}$ with $FGM \cong M$ for all $M \in \mathbb{A}$);*
uniform retraction: *if it is a retraction, and every splitting of its arrow part refines to a right inverse (more precisely, if $\Gamma : |\mathbb{A}| \longrightarrow |\mathbb{B}|$ is such that $F\Gamma M \cong M$, $M \in \mathbb{A}$ then there is a functor $G : \mathbb{A} \longrightarrow \mathbb{B}$, where GM is a quotient of ΓM and $FGM \cong M$);*
(co)reflection: *if it has a right inverse right (resp. left) adjoint.*

Lemma 1. *A powerful and subobject covering functor is essentially surjective.*

Lemma 2. *$F : \mathbb{A} \longrightarrow \mathbb{B}$ is faithful if and only if*

$$F(\varphi) \vdash F(\psi) \Longrightarrow \varphi \vdash \psi \tag{1}$$

As the converse of (1) is always true, a faithful coherent functor F always induces an "order isomorphism" on the subobject lattices.

To *prove* lemma 2, use the fact that $\varphi \vdash \psi$ if and only if $\varphi = \varphi \wedge \psi$.

Proposition 1. *A coherent functor must be full as soon as it is both faithful and powerful.*

Proof. Since F is powerful, the graph of any $h : FA \longrightarrow FA'$ must be in the essential image of F: there must be a monic $\kappa \rightarrowtail A \times B$ in $\mathbb{A}$ the F-image of which is isomorphic with the graph $\chi = \langle \mathrm{id}, h \rangle : FA \longrightarrow FA \times FB$. The relation $F\kappa$ thus satisfies;

$$\delta_{FA} \vdash F\kappa \; ; F\kappa^{op}$$
$$F\kappa^{op} \; ; F\kappa \vdash \delta_{FB}$$

which respectively tell that it is total and single valued. Taking into account that for the identity relation $\delta = \langle \mathrm{id}, \mathrm{id} \rangle$ holds $\delta_{FX} = F\delta_X$, and using (1), we conclude that κ is a total and single valued relation in $\mathbb{A}$. In any regular category, such a relation must be isomorphic to one in the form $\langle \mathrm{id}, k \rangle : A \longrightarrow A \times B$. Since clearly $F\langle \mathrm{id}, k \rangle = \langle \mathrm{id}, h \rangle$, we conclude that $Fk = h$. $\qquad \square$

---

[5] Recall that subobjects are *isomorphism classes* of monics.

3.2 Basic results

In the sequel, we assume that $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a coherent functor between coherent categories, and $F^{\#} : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ is the functor induced by the precomposition. We use and extend some results from [21]. Note that some of them essentially depend on strong model theoretic assumptions, such as compactness. The proofs are thus largely non-constructive, as they depend on the axiom of choice.

Proposition 2. *F is faithful if and only if $F^{\#}$ is essentially surjective.*

Proof. By lemma 2, F is faithful if and only if

$$F\varphi \vdash F\psi \Longleftrightarrow \varphi \vdash \psi$$

By the completeness theorem [21, thm. 5.1.7] $F\varphi \vdash F\psi$ holds if and only if

$$\forall N \in \mathsf{Mod}(\mathbb{B}).\ NF\varphi \subseteq NF\psi$$

whereas $\varphi \vdash \psi$ holds if and only if

$$\forall M \in \mathsf{Mod}(\mathbb{A}).\ M\varphi \subseteq M\psi$$

The last two statements are clearly equivalent if $F^{\#}$ is essentially surjective, i.e.

$$\forall M \in \mathsf{Mod}(\mathbb{A}) \exists N \in \mathsf{Mod}(\mathbb{B}).\ M \cong F^{\#}N$$

Conversely, if there is $M \in \mathsf{Mod}(\mathbb{A})$ different from $F^{\#}N$ for all $N \in \mathsf{Mod}(\mathbb{B})$, one can use compactness to construct a formula ψ such that $NF\psi$ is true for all models N of $\mathbb{B}$, whereas $M\psi$ is not. $\qquad\square$

Definition 2. *$F^{\#} : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ is said to be* subfull *if every $\mathbb{A}$-homomorphism $h : F^{\#}N' \longrightarrow F^{\#}N''$ preserves all $\mathbb{B}$-subobjects, i.e. for every monic $D \overset{d}{\rightarrowtail} FA$ in $\mathbb{B}$ holds*

$$hA(N'D) \subseteq N''D$$

$$(2)$$

Proposition 3. *F is powerful if and only if $F^{\#}$ is subfull.*

Proof. By definition, F is powerful if and only if every $D \overset{d}{\rightarrowtail} FA$ is in the essential image of F, i.e. $d \cong Fs$ for some $S \overset{s}{\rightarrowtail} A$. So (2) must commute because it is isomorphic with the square

$$
\begin{array}{ccc}
N'FS & \xrightarrow{\;\;hS\;\;} & N''FS \\
\Big\downarrow{\scriptstyle N'Fs} & & \Big\downarrow{\scriptstyle N''Fs} \\
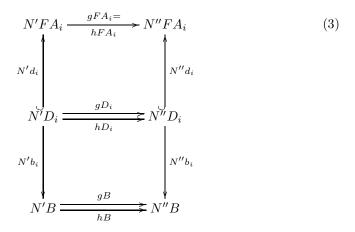N'FA & \xrightarrow[\;\;hA\;\;]{} & N''FA
\end{array}
$$

which commutes by the naturality of h.

The other way around, the fact that the subfullness of $F^{\#}$, i.e. the commutativity of squares (2) implies that F is powerful is one of the main constituents of the Makkai-Reyes conceptual completeness theorem [21, ch. 7§1]. The proof can be extracted from [21, thms. 7.1.4–4'], and essentially depends on compactness. □

Proposition 4. *F is subcovering if and only if $F^{\#}$ is faithful.*

Proof. Suppose F is subcovering and let $F^{\#}g = F^{\#}h$ for some $\mathbb{B}$-homomorphisms $g, h : N' \longrightarrow N''$. The equation $F^{\#}g = F^{\#}h$ means that $gFA = hFA : N'FA \longrightarrow N''FA$ for all $A \in \mathbb{A}$.

I claim that then $gB = hB : N'B \longrightarrow N''B$ must hold for every $B \in \mathbb{B}$. Since F is subcovering, for each B there is a finite diagram D, with (1) a colimit cocone to B, i.e. a jointly epimorphic family $\{D_i \overset{b_i}{\twoheadrightarrow} B\}_{i=1}^n$, and (2) the inclusions $\{D_i \overset{d_i}{\rightarrowtail} FA_i\}_{i=1}^n$ for some objects $A_1, \dots A_n \in \mathbb{A}$. Hence

$$
\begin{array}{ccc}
N'FA_i & \overset{gFA_i=}{\underset{hFA_i}{\xrightarrow{\hspace{1.2cm}}}} & N''FA_i \\[4pt]
\Big\uparrow{\scriptstyle N'd_i} & & \Big\uparrow{\scriptstyle N''d_i} \\[4pt]
N'D_i & \overset{gD_i}{\underset{hD_i}{\rightrightarrows}} & N''D_i \\[4pt]
\Big\downarrow{\scriptstyle N'b_i} & & \Big\downarrow{\scriptstyle N''b_i} \\[4pt]
N'B & \overset{gB}{\underset{hB}{\rightrightarrows}} & N''B
\end{array}
\qquad\qquad (3)
$$

Naturality of g and h now yields

$$N''d_i \circ gD_i = gFA_i \circ N'd_i$$
$$= hFA_i \circ N'd_i$$
$$= N''d_i \circ hD_i$$

But since models are left exact, each $N''d_i$ is still a monic, and therefore $gD_i = hD_i$, for all $i = 1, \ldots, n$.

Using naturality again, we get

$$gB \circ N'b_i = N''b_i \circ gD_i$$
$$= N''b_i \circ hD_i$$
$$= hB \circ N'b_i$$

But since models preserve the finite unions of subobjects $\{N'b_i\}_{i=1}^n$ must be jointly monic again, and therefore $g_B = h_b$. Thus $g = h$, and $F^\#$ is faithful.

For the converse, one assumes that there is $B \in \mathbb{B}$ not subcovered by F, and, using compactness, constructs models N' and N'' and two homomorphisms $g \neq h : N' \longrightarrow N''$ such that $F^\# g = F^\# h$. The details are in [21, thms. 7.1.6–6']. □

Logical meaning. Proposition 2 tells that each $\mathbb{A}$-model extends back along $F^\#$ to some $\mathbb{B}$-model if and only if $F : \mathbb{A} \longrightarrow \mathbb{B}$ is faithful. However, this does not guarantee that every $\mathbb{A}$-homomorphism between $\mathbb{A}$-models will extend to a $\mathbb{B}$-homomorphism between their extensions. Indeed, according to proposition 3, a necessary condition for this is that $F : \mathbb{A} \longrightarrow \mathbb{B}$ is powerful.

Together, these conditons provide a basis for aligning syntactical and the semantical definitions of parametricity, as described in section 2.4.

4 Characterizing parametric specifications

Theorem 1. *For a coherent functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ and the induced "reduct" $F^\# : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$, the following statements are equivalent.*

 (a) *F is a powerful embedding.*

 (b) *$F^\#$ is subfull and essentially surjective.*

 (c) *$F^\#$ is a uniform retraction.*

If $\mathsf{Mod}(\mathbb{B})$ has coproducts, then the above conditions are also equivalent with

 (d) *$F^\#$ is coreflection.*

Note that, since $\mathsf{Mod}(\mathbb{B})$ is finitely accessible, it has coproducts if and only if it is locally finitely presentable, i.e. when $\mathbb{B}$ classifies an essentially algebraic theory [1, sec. 3D].

Proof. **(a)**⇔**(b)** By proposition 1, it suffices to check that F is faithful and powerful. By proposition 2, F faithful if and only if $F^\#$ is essentially surjective. By proposition 3, F is powerful if and only if $F^\#$ is subfull.

To simplify the proof of **(b)**⇒**(c)**, we shall freely use the established equivalence **(a)**⇔**(b)**. Given that $F^\#$ is essentially surjective and subfull, we thus know that F is full, faithful and powerful. Using all that, we define $\Phi : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$, such that $F^\# \circ \Phi \cong \mathrm{Id}$.

Since $F^\#$ is essentially surjective, for every M in $\mathsf{Mod}(\mathbb{A})$, there is some L in $\mathsf{Mod}(\mathbb{B})$ such that $M \cong F^\#L$. But the homomorphisms to or from M may not extend to every such L, so we cannot simply take $\Phi M = L$.

On the other hand, like any functor, $M : \mathbb{A} \longrightarrow \mathsf{Set}$ has the right Kan extension, a functor $F_\# M : \mathbb{B} \longrightarrow \mathsf{Set}$ [18], defined

$$F_\# M(B) = \varprojlim M \circ \mathsf{Cod}\,(B/F) \tag{4}$$

where B/F is the comma category, spanned by the arrows in the form $B \xrightarrow{a} FA$ in $\mathbb{B}$. A morphism from $B \xrightarrow{a} FA$ to $B \xrightarrow{c} FC$ is an arrow $g : A \longrightarrow C$ in $\mathbb{A}$ such that $Fg \circ a = c$. The image of $B \in \mathbb{B}$ along $F_\# M$ is thus the limit of the diagram $B/F \xrightarrow{\mathsf{Cod}} \mathbb{A} \xrightarrow{M} \mathsf{Set}$.

The construction $F_\#$ is functorial and it is not hard to see that $F^\# \circ F_\# \cong \mathrm{Id}$ holds if and only if F is faithful. So $F_\# M$ might be a candidate for ΦM. But the assumption that $M : \mathbb{A} \longrightarrow \mathsf{Set}$ is coherent does not generally follow for $F_\# M : \mathbb{B} \longrightarrow \mathsf{Set}$. The $F_\#$-image of an $\mathbb{A}$-model M may not be a $\mathbb{B}$-model, and the functor $F_\# : \mathsf{Set}^\mathbb{A} \longrightarrow \mathsf{Set}^\mathbb{B}$ does not restrict to a functor $\mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$.

But the desired model $\Phi M : \mathbb{B} \longrightarrow \mathsf{Set}$ can actually be "interpolated" between the Kan extension $F_\# M : \mathbb{B} \longrightarrow \mathsf{Set}$, and the arbitrary model $L : \mathbb{B} \longrightarrow \mathsf{Set}$ such that $F^\# L \cong M$.

First of all, since $F^\# \dashv F_\#$, every $F^\# L \longrightarrow M$ induces $L \longrightarrow F_\# M$. Given, as above $M \cong F^\# L$, for every $a : B \longrightarrow FA$ in $\mathbb{B}$, there is $La : LB \longrightarrow LFA \cong MA$. Hence a cone $\langle La \rangle_{a \in B/F} : LB \longrightarrow M \circ \mathsf{Cod}\,(B/F)$. By definition (4), this cone induces a unique arrow $\phi B : LB \longrightarrow F_\# M(B)$.

Let the functor $\Phi M : \mathbb{B} \longrightarrow \mathsf{Set}$ be defined as the monic image of $\phi : L \longrightarrow F_\# M$, i.e.

$$\phi B \; : \; LB \longrightarrow\!\!\!\!\!\longrightarrow \Phi M(B) \lhook\joinrel\longrightarrow F_\# M(B) \tag{5}$$

This ΦM will indeed be a model. Although $F_\# M : \mathbb{B} \longrightarrow \mathsf{Set}$ is not a model, when $F : \mathbb{A} \longrightarrow \mathbb{B}$ and $M : \mathbb{A} \longrightarrow \mathsf{Set}$ preserve (finite) limits, then $F_\# M : \mathbb{B} \longrightarrow \mathsf{Set}$ weakly preserves them: for every (finite) diagram $\Delta : I \longrightarrow \mathbb{B}$, the set $F_\# M(\varprojlim \Delta)$ is a weak limit of $F_\# M(\Delta)$ and thus contains $\varprojlim F_\# M(\Delta)$ as a retract.

Together with the coherence of $L : \mathbb{B} \longrightarrow \mathsf{Set}$, this weak preservation property of $F_\# M$ suffices for the coherence of $\Phi M : \mathbb{B} \longrightarrow \mathsf{Set}$. E.g., it preserves the

products because the map from $\Phi M(B) \times \Phi M(D)$ to $\Phi M(B \times D)$ on

$$
\begin{array}{ccccc}
LB \times LD & \twoheadrightarrow & \Phi M(B) \times \Phi M(D) & \hookrightarrow & F_\# M(B) \times F_\# M(D) \qquad (6) \\
\downarrow \cong & & \vdots & & \uparrow \\
L(B \times D) & \twoheadrightarrow & \Phi M(B \times D) & \hookrightarrow & F_\# M(B \times D)
\end{array}
$$

must be both surjective and injective.

The object part of $\Phi : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$ is thus determined by (5). Notice that Φ is not unique, as the definition depends on the choice of L, $F^\# L \cong M$.

To define the arrow part of Φ, take an arbitrary $\mathbb{A}$-homomorphism $h : M' \longrightarrow M''$. It surely induces a natural transformation $F_\# h : F_\# M' \longrightarrow F_\# M''$, and we can find $\mathbb{B}$-models L' and L'' that map by $F^\#$ to M' and M'', and determine $\mathbb{B}$-models $\Phi M'$ and $\Phi M''$; but $h : M' \longrightarrow M''$ in general does not lift to a homomorphism $L' \longrightarrow L''$. However, $\Phi h : \Phi M' \longrightarrow \Phi M''$ can be derived from $F_\# h : F_\# M' \longrightarrow F_\# M''$ alone.

To simplify notation, write $N' = \Phi M'$ and $N'' = \Phi M''$ and $k = \Phi h$ for the desired homomorphism.

We are given a natural family $hA : M'A \longrightarrow M''A$ and we want to extend it to $kB : N'B \longrightarrow N''B$, so that $kFA = hA$. In other words, we have the subfamily of functions $kFA : N'FA \longrightarrow N''FA$, $A \in \mathbb{A}$, and we need to complete it to a natural family $kB : N'B \longrightarrow N''B$, $B \in \mathbb{B}$.

Consider first, for each $B \in \mathbb{B}$, the set $\mathcal{E}_B$ of regular epimorphisms $e : B \twoheadrightarrow FA_e$ in $\mathbb{B}$. The e-th component of the limit cone $F_\# M(B) \longrightarrow M \circ \mathsf{Cod}(B/F)$ is a function $F_\# M(B) \longrightarrow MA_e$. Hence the map

$$
F_\# M(B) \longrightarrow \prod_{e \in \mathcal{E}_B} MA_e \qquad (7)
$$

Since $F : \mathbb{A} \longrightarrow \mathbb{B}$ is powerful, this map is injective. By postcomposing (5) with it, one gets

$$
\langle Le \rangle_{e \in \mathcal{E}_B} \; : \; LB \twoheadrightarrow \Phi M(B) \hookrightarrow \prod_{e \in \mathcal{E}_B} LFA_e \qquad (8)
$$

because $MA_e = LFA_e$. Of course, since L is coherent, each $Le : LB \longrightarrow LFA_e$ is a surjection. The set $\Phi M(B)$ can thus also be obtained by taking the product of all sets LFA_e, such that there is some regular epi $e : B \longrightarrow FA_e$ in $\mathbb{B}$, and then extracting from this product the image of the tuple formed by all epis $Le : LB \twoheadrightarrow LFA_e$.

The construction of $kB : N'B \longrightarrow N''B$ now proceeds by the following steps:

(i) define a function

$$
\kappa B : N'B \longrightarrow \wp(N''B)
$$

such that

$$\kappa FA(x) = \{hA(x)\}$$

(ii) show that $\kappa B(x)$ is nonempty for every $x \in N'B$;

(iii) show that $\kappa B(x)$ has at most one element for every $x \in N'B$; writing $kB(x)$ for the only element of $\kappa B(x)$, we get the function $kB : N'B \longrightarrow N''B$;

(iv) prove that the obtained family $kB : N'B \longrightarrow N''B$, $B \in \mathbb{B}$ is natural, i.e. forms $k : N' \longrightarrow N''$.

(i) Using the same set $\mathcal{E}_B$ of regular epis $e : B \longrightarrow\!\!\!\!\rightarrow FA$ as above, define
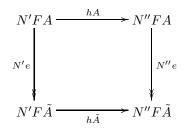
$$\kappa^e B(x) = (N''e)^{-1} \circ hA \circ N'e(x)$$
$$\kappa B(x) = \bigcap_{e \in \mathcal{E}_B} \kappa^e B(x)$$

For $B = FA$, $\kappa^{\mathrm{id}} FA(x) = \{hA(x)\}$. Moreover, for every $e \in \mathcal{E}_{FA}$ holds

$$\kappa^{\mathrm{id}} FA(x) \subseteq \kappa^e FA(x) \tag{9}$$

Indeed, since F is full, the naturality of h implies that the square

$$
\begin{array}{ccc}
N'FA & \xrightarrow{\ hA\ } & N''FA \\
{\scriptstyle N'e}\Big\downarrow & & \Big\downarrow{\scriptstyle N''e} \\
N'F\tilde{A} & \xrightarrow[\ h\tilde{A}\ ]{} & N''F\tilde{A}
\end{array}
$$

commutes. Hence (9), and thus $\kappa FA(x) = \{hA(x)\}$, as asserted.

(ii) For every $B \in \mathbb{B}$, the set $\mathcal{E}_B$ is nonempty because it surely contains the regular epi part $B \twoheadrightarrow FI \rightarrowtail F1 \cong 1$. $F1$ is terminal because F is coherent; the regular image of $B \longrightarrow F1$ is in the image of F because F is powerful.

Moreover, since N'' is coherent, and $B \longrightarrow\!\!\!\!\rightarrow FI$ is a cover (regular epi) $N''B \longrightarrow N''FI$ must be a surjection. So if $N''B$ is empty, $N''FI$ must be empty, and hence $N'FI$ must be empty, because there is a function $hI : N'FI \longrightarrow N''FI$. But there is also a function $N'B \longrightarrow N'FI$, and thus $N'B$ must be empty as well, so there is a unique $kB : N'B \longrightarrow N''B$, and we are done.

With no loss of generality, we can thus assume that $N''B$ is nonempty. Since $N''e : N''B \longrightarrow NFA$, $e \in \mathcal{E}_B$, is a surjection, all NFA are nonempty, and *moreover*, every $\kappa^e(x) = (N''e)^{-1} \circ hA \circ N'e(x)$ is nonempty.

Finally, for any $e_0 : B \longrightarrow\!\!\!\!\rightarrow FA_0$ and $e_1 : B \longrightarrow FA_1$ from $\mathcal{E}_B$ the intersection $\kappa^{e_0} \cap \kappa^{e_1}$ is nonempty as well. Toward a proof, consider the pair $\langle e_0, e_1 \rangle : B \longrightarrow FA_0 \times FA_1 \cong F(A_0 \times A_1)$ in $\mathbb{B}$. Factoring, and using once again the assumption

that F is powerful, we get $e_2 : B \longrightarrow FA_2$, with a pair $\langle p_0, p_1 \rangle : A_2 \longrightarrow A_0 \times A_1$ in $\mathbb{A}$ such that $e_i = Fp_i \circ e_2$, $i = 0, 1$. But $N''e_i = N''Fp_i \circ N''e_2$ implies

$$\kappa^{e_2}(x) \subseteq \kappa^{e_0}(x) \cap \kappa^{e_1}(x)$$

for all $x \in N'B$. Since $\kappa^{e_2}(x)$ has been proved nonempty, $\kappa^{e_0}(x) \cap \kappa^{e_1}(x)$ is.

A similar reasoning applies to any finite intersection of κ^es. But for the quotients $e \in \mathcal{E}_B$ in a coherent category $\mathbb{B}$ the compactness applies: if any finite family is consistent, then the whole set together is. Therefore, $\kappa B(x)$ is nonempty.

(iii) So we can surely chose $kB(x) \in \kappa B(x)$. No matter which element we choose, the equation
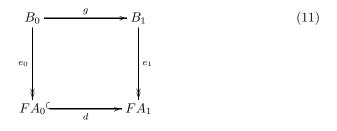
$$N''e \circ kB = kFA \circ N'e \tag{10}$$

will hold for every $e \in \mathcal{E}_B$, because $kFA = hA$ and the definition of κB implies

$$N''e \circ \kappa B = hA \circ N'e$$

On the other hand, recall that $N''B = \Phi M''B$ was defined so as to make the function $\langle N''e \rangle_{e \in \mathcal{E}_B}$ injective. But this means that the set of equations (10), for all $e \in \mathcal{E}_B$, together determine at most one $kB(x)$, since the functions $N''e$ are jointly injective.
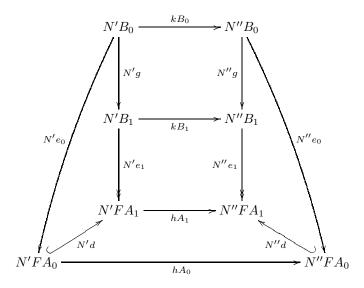
So the family $hA : N'FA \longrightarrow N''FA$, $A \in \mathbb{A}$, extends to a uniquely determined family $kB : N'B \longrightarrow N''B$, $B \in \mathbb{B}$.

(iv) To prove that the family $kB : N'B \longrightarrow N''B$ is natural, take an arbitrary arrow $g : B_0 \longrightarrow B_1$ in $\mathbb{B}$ and an arbitrary $e_1 : B_1 \longrightarrow FA_1$ from $\mathcal{E}_{B_1}$. Let e_0 be the coimage of $e_1 \circ g$

$$\begin{array}{ccc} B_0 & \xrightarrow{\ \ g\ \ } & B_1 \\ \downarrow{\scriptstyle e_0} & & \downarrow{\scriptstyle e_1} \\ FA_0 & \xrightarrow{\ \ d\ \ } & FA_1 \end{array} \tag{11}$$

The codomain of e_0 is in the image of F because it is powerful.

We want to prove that the upper square in the diagram

commutes. The lower square and the large outside trapezoid surely commute by the definition of kB. The small trapezoid commutes by the naturality of h, and the two triangles simply as the images of (11). Chasing, one concludes that

$$N''e_1 \circ kB_1 \circ N'g = N''e_1 \circ N''g \circ kB_0$$

But e_1 was taken as an arbitrary element of $\mathcal{E}_{B_1}$, so the last equation holds for all such. Since they are, by the construction of $N'' = \Phi M''$, jointly monic,

$$kB_1 \circ N'g = N''g \circ kB_0$$

follows.

This completes the proof of $(\mathbf{b}) \Rightarrow (\mathbf{c})$. The converse $(\mathbf{c}) \Rightarrow (\mathbf{b})$ can be proved by modifying [21, thm. 7.1.4–4']. The argument is lengthy, based on the Los-Tarski theorem, and I do not see a way to improve on it, so the reader may wish to consult the original.

Finally, to connect $(\mathbf{d})$ with the other three conditions, note that the assumption of coproducts makes $\mathsf{Mod}(\mathbb{B})$ into a locally finitely presentable category, so that $F^{\#}$ must have a left adjoint, like in [10, § 5], obtained by restricting the left Kan extension of F. Hence $(\mathbf{d}) \Leftrightarrow (\mathbf{a})$. But a proof of this was already in [9] and [14], albeit in a slightly different setting. $\square$

An immediate consequence of theorem 1 and proposition 4 is a precise syntactic characterisation of *definitional extensions*, the interpretations F which induce an equivalence $F^{\#}$ between the model categories. The class is essentially larger than assumed in any of the implemented versions.

Corollary 1. $F^{\#} : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ *is an equivalence if and only if* $F : \mathbb{A} \longrightarrow \mathbb{B}$ *is a powerful embedding, and subcovering.*

A proof of this can also be derived from Makkai-Reyes' *conceptual completeness* theorem [21, thm. 7.1.8], which is the main result of their book.

5   Conclusions and further work

The research reported in this paper was originally motivated by the questions arising from the semantics and the usage of SPECWARE$^{\text{TM}}$, a tool for the automatic synthesis of software systems, developed at Kestrel Institute. In particular, the original semantics of pspecs as an abstract family of arrows [25] needed to be refined into a precise syntactic characterisation and verified semantically. This task took us far afield, into nontrivial model theory and functorial calculus, and brought about the above theorem relating two extant notions of parametricity. As suggested at the end of section 2.4, it can be viewed as an indexed completeness result. Formalizing this view might lead to various conceptual and meta-theoretical insights.

But the question of the practical repercussions of the presented material, or of their absence, seems even more interesting. The immediate task should probably be to analyze closely related families of coherent functors, capturing the instantiations and the implementations of theories. The practice of parametric specification is based upon them as much as upon the family of pspecs, studied in the present paper. Some important issues of refinement directly require this further analysis.

However, as we are not very far in any of these tasks, the main point of presenting this work currently is not this or that particular result, but showing categorical model theory at work in the software specification framework and suggesting a first step or two toward developing specific tools for analyzing and designing specification frameworks.

If, as is often stated, the increasing complexities and dynamics of evolving software development tasks make semantical analyses increasingly important, even indispensable in critical cases, then mathematical methods of the kind presented here may come to play an increasingly important role, as they may provide enough abstraction to resolve the concrete problems where formal methods are genuinely needed.

References

1. J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*. LMS Lect. Note Series 189 (Cambridge Univ. Press 1994)

2. J. Adámek, P.T. Johnstone, J.A. Makowsky and J. Rosický, Finitary sketches, *J. Symbolic Logic* 62(1997) 699–707

3. M. Barr and C. Wells, *Category Theory for Computing Science.* (Prentice Hall 1990)

4. R. Burstall and J. Goguen, The semantics of Clear, a specification languge. In: D. Bjorner, ed., *Proceedings, 1979 Copenhagen Winter School on Abstract Software Specification.* Lecture Notes in Computer Science 86 (Springer 1980)

5. Th. Dimitrakos, *Formal Support for Specification Design and Implementation.* Thesis (Univ. of London 1998)

6. H. Ehrig and M. Große-Rhode, Functorial theory of parametrized specifications in generalized specification framework, *Theoret. Comput. Sci.* 135(1994) 221–266

7. H. Ehrig and H.J. Kreowski, Parameter passing commutes with implementation of parametrized data types. In [23] 197–211

8. H. Ehrig and B. Mahr, *Fundamentals of Algebraic Specification 2. Module Specifications and Constraints.* EATCS Monographs in Theor. Comp. Sci., vol. 21 (Springer 1990)

9. H. Ganzinger, Parametric specifications: parameter passing and implementations with respect to observability. *ACM Trans. on Prog. Lang. and Syst.* 5(1983) 318–354

10. P. Gabriel and F. Ulmer, *Lokal Präsentierbare Kategorien.* Lecture Notes in Mathematics 221 (Springer 1971)

11. J. Goguen and R.M. Burstall, CAT, a system for the structured elaboration of correct programs from structured specifications. Tech. report CSL-118 (SRI 1980)

12. J. Goguen, Parametrized programming. *Trans. on Software Engineering* 10-5(1984) 528–543

13. J. Goguen, Principles of parametrized programming. In: *Software Reusability. Vol I: Concepts and Models.* (Addison-Wesley 1989) 159–225

14. J. Goguen and J. Meseguer, Universal realization, persistent interconnection and implementation in abstract modules. In [23] 265–281

15. J. Goguen, J. Meseguer and D. Plaisted, Programming with parametrized abstract objects in OBJ. In: D. Ferrari et al., eds., *Theory and Practice of Software Technology* (North-Holland 1983) 163–193

16. F.W. Lawvere, *Functorial Semantics of Algebraic Theories.* Thesis (Columbia Univ. 1963)

17. M. Löwe and U. Wolter, Parametric algebraic specifications with Gentzen formulas — from quasi-freeness to free functor semantics. *Math. Structures Comput. Sci.* 5(1995) 69–111

18. S. Mac Lane, *Categories for the Working Mathematician.* Graduate Texts in Mathematics 5 (Springer 1971)

19. M. Makkai, Stone duality for first order logic. *Advances in Math.* 65(1987)

20. M. Makkai and R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory.* Contemporary Mathematics 104 (AMS 1989)

21. M. Makkai and G. Reyes, *First Order Categorical Logic.* Lecture Notes in Mathematics 611 (Springer 1977)

22. E. Manes, *Algebraic Theories.* (Springer 1976)

23. M. Nielsen and E.M. Schmidt, eds., *Proceedings of Ninth ICALP.* Lecture Notes in Computer Science 140 (Springer 1982)

24. D. Sanella and M. Wirsing, Implementation of parametrized specifications. In [23] 473–488

25. Y.V. Srinivas, Refinement of parametrized algebraic specifications. In: R. Bird and L. Meertens, eds., *Algorithmic Languages and Calculi*. (Chapman & Hall 1997) 164–186

26. C. Strachey, Fundamental concepts in programming languages, *unpublished lecture notes*, 1967

27. J.W. Thatcher, E.G. Wagner and J.B. Wright, Data type specification: parametrization and the power of specification techniques. *ACM Trans. of Prog. Lang. and Syst.* 4(1982) 711–732 (earlier version: *Proc. of 10th SIGACT Symp. on Theory of Computing* (1978) 119–132)

28. R. Waldinger, L. Blaine, D. Espinosa, L.-M. Gilham, A. Goldberg, C. Green, R. Jüllig, J. Liu, J. McDonald, D.R. Smith, Y.V. Srinivas, T.C. Wang and S. Westfold, SPECWARE^{TM} *Language Manual* (Kestrel Institute 1998) — available upon request from `keep@kestrel.edu`